A Closed-loop Approach to Dynamic Assortment Planning

Esra Çınar* Victor Martínez-de-Albéniz†

Submitted: July 10, 2014.

Abstract

Firms are constantly trying to keep the customers interested by refreshing their assortments. In industries such as fashion retailing, products are becoming short-lived and, without product introductions or in-store novelties, category sales quickly decrease. We model these dynamics by assuming that products lose their attractiveness over time and we let the firm enhance the assortment at a cost, for single or multiple categories. We characterize the optimal closed-loop policy that maximizes firm profits. When adjustment costs are linear in the attractiveness, we find that an assort-up-to policy is best: it is optimal to increase category attractiveness to a target level, which is independent of the current attractiveness. Interestingly, when there is no constraint on the maximum achievable attractiveness for any category, it is optimal to invest in one category only, in the extreme cases single period and infinite horizon. In that case the optimal assort-up-to levels can be characterized in closed-form. In contrast, with capacity constraints or multi-period finite horizons, it is optimal to diversify the investment across categories. Beyond our structural results, we study the value of our closed-loop approach compared to open-loop and front-loaded strategies. We find that our policies can provide significant value (up to 10% of additional profits), especially when the costs are not too high nor too low, and there is significant uncertainty about the decay rate of products, as in most fashion retailing contexts.

1 Introduction

In many industries, the length of product life-cycles has been steadily decreasing. This phenomenon is most acute in apparel retailing, in which firms that used to release two collections every year are now experiencing many more cycles, e.g., Esprit launches twelve mini-collections

*IESE Business School, University of Navarra, Barcelona, Spain, ecinar@iese.edu
†IESE Business School, University of Navarra, Barcelona, Spain, valbeniz@iese.edu

per year (Esprit 2011). This product release hyper-activity is triggered by the fact that fashion-forward customers pay more attention to the latest trends and older products become less attractive. Hence, to keep the customers interested, retailers have to change or alter the products they offer more often. Similarly, in consumer electronics retailing, the phone or tablet categories exhibit continuous new product introductions from different brands. There is no significant technological advancement in the functions of the new releases, but nevertheless consumers do prefer the latest products. For books, music or movies, novelty also provide a boost on popularity: stores will provide special, better displays for these items; movie theaters will be even more drastic, removing movies from the offer after a few weeks.

To cope with these dynamic changes on the consumer side, retailers react with dynamic store policies, including new product introductions, dynamic in-store merchandizing or guidance of salespeople efforts towards certain products (via incentives). In the not so distant past, companies used to have limited ability to revise their decisions during their selling season because there was little visibility on market trends, and in any case implementing store changes was too slow. However, with the adoption of information technologies and postponement and quick-response processes, it is nowadays possible to leverage real-time demand to dynamically adjust what happens at the store level. We call these dynamic actions dynamic assortment decisions, although they encompass not only choices of which products to sell in the store, but all the other relevant variables that may influence product, category and store attractiveness as well.

An example is the *fast fashion* model adopted by some fashion apparel retailers. It is essentially based on the effective use of information obtained early in the season to quickly manufacture new products for sale later in the season. In particular, the information is used to produce and distribute items to the stores based on refined demand forecasts, but also to create new products similar to successful products (e.g., by offering more colors of a certain model), thereby managing variety and assortment size dynamically. Zara and H&M are good illustrations of this behavior. They use frequent assortment rotation to generate store traffic and increase sales (Caro and Martínez-de-Albéniz 2009, 2012). For instance, Zara and other fashion companies employ an attribute-based in-season design strategy: they learn the attributes of the high-selling products during the season and design new products with those attributes. If a certain color or fabric is selling well, the assortment can be renewed so as to give more weight to the popular fabric-color combinations. In a way, one could talk about *assortment replenishment*, as opposed to inventory replenishment, since what the retailer does is increase the assortment attractiveness when new information becomes available.

In this paper, we study the dynamic assortment planning problem of a retailer that wants to maximize its profits during the season. The retailer carries items grouped in several categories, fixed over time, and we use a logit demand model to take into account substitution across different categories, as well as a possible outside option. One essential ingredient of our model is that category attractiveness may decay over time, due to changes in consumers' interest during the season. The retailer can compensate this loss of attractiveness by putting an effort in the category, which will help it increase sales and revenues. This can be seen as an investment in new, fresher, products, or any other in-store merchandizing policy that can boost the category's perception. Of course, the cost associated with such investment should yield a return later during the selling season.

We propose a dynamic programming formulation of the problem. In every period, e.g., every week, the retailer observes the latest information regarding category attractiveness and can choose to increase attractiveness at a cost, possibly within a limited range. The objective is to maximize discounted profits in expectation, by using the most appropriate closed-loop policy. Such closed-loop approach is more suitable than an open-loop policy, where all decisions have to be made at the beginning of the planning horizon. Specifically, in fashion products, there is significant uncertainty about the performance of products, i.e., their attractiveness over time, so closed-loop policies can perform much better than open-loop ones.

Our main results are the following. We characterize the optimal closed-loop policy for single and multiple category problems under the assumption that the costs of increasing attractiveness are linear. We find that an assort-up-to policy is optimal when margins are identical across categories. Namely, the optimal policy is to increase category attractiveness up to a target level, which is a parameter to be determined, independent of the starting attractiveness. When margins are different across categories, the problem is generally not tractable. However, we can still obtain strong results in two extreme cases: when there is a single period (short horizon) or infinite number of periods (very long horizon). In such scenarios, it is best to choose only one category for investment, as long as there is no constraint on the maximum achievable attractiveness for any category. In addition, the optimal assort-up-to levels can then be characterized in closed-form, and depend on the category price, cost and expected decay. In contrast, when the attractiveness of any category is subject to a capacity constraint, then the optimal solution involves diversification across multiple categories, and we describe efficient algorithms to find the optimal set of categories in which to invest, even when margins differ across categories.

In addition to our analytical results, we numerically compare the closed-loop approach that we propose to (i) an open-loop policy to evaluate the *value of responsiveness*, that is, the benefits of updating the decisions after observing the attractiveness state of the assortment; and to (ii) a front-loaded policy to evaluate the *value of novelty*, that is, the benefits of releasing attractiveness in the middle of the season as opposed to just at the beginning. We find that our closed-loop policies can provide a relatively interesting increase in profits, the value of which depends on the problem parameters. An increase of more than +10% can be found when the costs are not too high nor too low, and when there is significant uncertainty about the decay rate of products, as in the fashion retailing contexts mentioned above. Our results also provide some critical insight on the value of dynamic assortments. Previous research (Caro et al. 2014) showed that when costs are high or expected decay is minimal, it is optimal to front-load all investments for an open-loop policy. However, if there is uncertainty involved, we show that there are still significant gains to be achieved by dynamically introducing attractiveness in-season: it is the last-minute, closed-loop aspect that delivers the value. This suggests that the value of dynamic assortments is highly sensitive to the timing of decision making.

The rest of the paper is organized as follows. We give an overview of the related literature in Section 2. In Section 3, the assortment planning model is described and some general properties are derived. A more detailed analysis of single-category and multi-category models is provided in Sections 4, 5 and 6. We compare closed-loop and open-loop policies in Section 7. We conclude in Section 8. All the proofs are given in a companion Appendix.

2 Literature Review

Our work is basically related to the literature of assortment planning problem of a retailer that carries seasonal products. We refer the reader to Kök et al. (2009) for a review of the assortment planning literature and industry practices. Most of the multi-period assortment planning studies do not consider dynamic settings. We focus this review on the papers that incorporate dynamic decisions into assortment planning.

2.1 Dynamic assortment planning with stationary demand

A class of dynamic assortment studies consider learning issues. Caro and Gallien (2007) is one of the first papers in this stream of work. They formulate an exploration versus exploitation trade-off problem. They develop a closed-form dynamic index policy to decide which assortment to offer every period. Ulu et al. (2012) also study learning about customer preferences in a locational model by dynamically adjusting the assortment. They show that it can be optimal to alternate between exploration and exploitation, and even offer assortments that lead to losses in order to gain information on consumer tastes. Rusmevichientong et al. (2010) consider an MNL choice model and construct an algorithm for optimizing dynamic assortments and parameter estimation, and determine the rate of convergence to the optimal policy. Sauré and Zeevi (2013) study assortment planning problems in which a more general random utility model drives consumer choice. They develop dynamic policies that balance the tradeoff between exploration and exploitation and prove that these policies satisfy some performance bounds. The latter two papers use adaptive learning approach, whereas Caro and Gallien (2007) and Ulu et al. (2012) use a Bayesian learning approach.

Other papers focus more on the revenue management aspect in retail: how a retailer should sell a limited amount of products over time. Bernstein et al. (2010) study how assortments for online retailing should be offered to the different customer segments, customized depending on the profile of inventory and remaining time in the selling season. They show that it may be optimal to "hide" some products to drive substitution to items with high inventory levels. Rusmevichientong and Topaloglu (2012) formulate a dynamic assortment optimization under the multinomial logit choice model where the parameters of the logit model are assumed to be unknown. The objective is to find an assortment that maximizes the worst-case expected revenue. They consider the problem of allocating a capacity to arriving customers over the planning horizon. This setting is the robust analogue of the single-leg airline seat allocation problem considered by Talluri and van Ryzin (2004), where the resource corresponds to the seat availability on the flight leg and the products correspond to different fare classes.

2.2 Dynamically evolving demand models

The papers mentioned above assume that the attractiveness of products in an assortment remain similar over time. There are a few papers that model explicitly the variation of consumer tastes over time. For instance, Caldentey and Caro (2010) model changing consumer preferences as a stochastic process called the "vogue", where the products within a category are differentiated by a single attribute (one dimensional location model). The retailer can choose to introduce products with new attributes that best fit the current consumer preference. The optimal dynamic assortment strategy is characterized, through the optimal time of product introduction.

Specific models that consider the reduction of a product's attractiveness over its life cycle have also been proposed, as we do in this paper. Conceptually, product interest decays with the time that they have spent in the assortment, as opposed to having consumers change taste randomly (implying that interest can go up or down). Caro et al. (2014) propose an attraction model in which product preference weights decay over time. They formulate the assortment packing problem, where given a collection a firm must decide in advance the release date of each product for the entire selling season. This model is quite close to ours, the main difference being that (i) we provide a closed-loop policy, instead of an open-loop one; and that (ii) we do not focus only on product introductions, which are integral in Caro et al. (2014), but generally consider effort, which may be done through product introductions or in-store marketing. Similar approaches have also been used to model movie sales: Krider and Weinberg (1998) and Ainslie et al. (2005) study attraction models where the market attractiveness of movies eventually decays over time as they age in the theaters. Ainslie et al. (2005) models box-office sales as a sliding window logit model with a gamma diffusion pattern and examine the factors that are likely to affect sales. Krider and Weinberg (1998) model the competition between two movies and conduct an equilibrium analysis of the product introduction timing game.

Finally, it is worth pointing out that there are also models where demand endogenously evolves as a function of past consumption (and thus of past retailer decisions). Caro and Martínez-de-Albéniz (2012) do not consider the decrease of attractiveness of products explicitly, but model the satiation effect of products on utility of variety-seeking consumers, who buys from multiple competing retailers. They determine the purchasing pattern of the customers and give an explicit expression of the consumption level in steady state: market shares take the form of an attraction model in which the attractiveness depends on price and product satiation. Honhon and Kök (2011) consider assortment choices explicitly with variety-seeking customers and identify cyclical patterns in the optimal assortment.

2.3 Demand driven by retailer effort

A broad stream of literature considering effort-dependent demands also exists in the operations literature. These models are usually studied when designing supply chain contracts. For example, Taylor (2002) includes a buyback option with the sales-rebate contract to coordinate the newsvendor with a fixed price but effort-dependent demand, while Krishnan et al. (2004) focus on promotional efforts and Cachon and Lariviere (2005) consider retailer effort in the revenue sharing contracts context. Wei and Chen (2011) characterize the structure of the optimal joint inventory and sales effort control policy where the inventory decision follows an (s, S) policy and then the sales effort is chosen depending on the inventory level. A particular type of effort comes from product display: Curhan (1973) and Corstjens and Doyle (1981) are two early papers that link shelf space to retail sales. Smith and Achabal (1998), Wang and Gerchak (2001), Balakrishnan et al. (2004), Caro et al. (2010) and Lago et al. (2013) discuss retail models where sales are increasing in inventory levels. All these papers do not consider product substitution, as we do here.

3 The Dynamic Assortment Planning Problem

3.1 Model formulation

Consider a retailer that offers N categories of products to its customers during a selling season. Although we call these categories for simplicity, they can really be items or product groups that remain stable over the season, even when their composition may change over time, e.g., T-shirts including different prints in different moments of the season. The model we propose can be adjusted to deal with both kinds of decision levels, i.e., whether only a single product is considered, which may be affected by in-store decisions, or a group of products is considered, which may be affected by new items entering the category. In our analysis, we call these categories to be able to give a sense of possible product groupings. The season is made of periods $t = 1, \dots, T$. T may be finite or infinite. Note that T can be considered as the number of revision points where the assortment can be refreshed, i.e., when its attractiveness can be increased.

Each category $i = 1, \dots, N$ has in each period $t = 1, \dots, T$ an attractiveness of y_{it}, which captures the interest that it generates on the consumers. There is also the outside option with an attractiveness of y_{0t}, which may vary over time if there is seasonality, possibly stochastically. We relate category attractiveness to its market share s_{it} through an attraction model:

$$s_{it} = \frac{y_{it}}{y_{0t} + \sum_{k=1}^{N} y_{kt}}.$$

This form is commonly used to model demand in marketing and operations management to capture assortment-based substitution (Kök et al. 2009).

After products are introduced into the store, their attractiveness decays naturally over time. We conceive the decay process of products as an intrinsic evolution driven by consumers' evolving preferences, which move away from previously known items. Indeed, studies from decision analysis suggest that happiness is derived from change itself, see Baucells and Sarin (2012).

We use a multiplicative decay to describe attractiveness dynamics. Namely, we assume that a category with an attractiveness y_{it} in period t would start period $t+1$ with an attractiveness $x_{it+1} := \varepsilon_{it+1} y_{it}$ for $i = 1, \ldots, N, t = 2, \ldots, T$. ε_{it} is an exogenous random variable between 0 and 1. We denote $\bar{\varepsilon}_{it}$ its average and $\sigma_{\varepsilon_{it}}$ its standard deviation. For simplicity, all our numerical results are shown with two-point distributions where $\bar{\varepsilon}_{it}$ and $\sigma_{\varepsilon_{it}}$ are varied. The attractiveness decay process is analogous to the decay in value of harvested fresh produce (Blackburn and Scudder 2009), and has been used before in retail applications (Caro et al. 2014).

Since the retailer wants to maintain the interest in the category, it will dynamically adjust attractiveness over time via promotions, advertising, new introductions, etc. Specifically, after observing the initial attractiveness level of a category, x_{it}, at the beginning of a given period the retailer is able to it can increase the attractiveness of each category, i.e., set $y_{it} \geq x_{it}$. We let $u_{it} := y_{it} - x_{it} \geq 0$ be the action taken by the retailer for category i in period t, and thus

$$y_{it} := \varphi_i(u_{it}, \varepsilon_{it}, y_{it-1}) \tag{1}$$

where $\varphi_i(u_{it}, \varepsilon_{it}, y_{it-1}) = u_{it} + \varepsilon_{it} y_{it-1}$. Putting an effort $\vec{u}_t := (u_{1t}, \ldots, u_{Nt})$ has a cost, denoted $C(\vec{u}_t)$. This formulation can be used to model different kinds of efforts. For example, u_{it} could be the attractiveness from new products introduced in the period, which we can consider to be a non-negative continuous variable for simplicity. Or it could be a binary variable representing an in-store marketing campaign on the category, through new fixtures, more sales support, etc. Although both situations can be accommodated in our model, we focus on the first one as it allows us to derive stronger analytical results. In other words, in the remainder of the paper, we assume that u_{it} is a non-negative continuous variable. We consider both cases where u_{it} can be constrained and unconstrained. The constraint comes from the assumption that the maximum attractiveness level a category can take may be limited. This might be due to a physical constraint, related to shelf space for instance. It might also be the case that the novelty that can be added to a category can be limited making the popularity of that category limited. We denote the maximum possible attractiveness level for category i as a_i, and we require that $a_i \geq y_{it}$. Thus, $u_{it} \leq a_i - x_{it}$. We denote $\vec{a} = (a_1, \ldots, a_N)$ be the vector of capacities for attractiveness.

Let $R(\vec{y}_t)$ be the revenue created by the vector of attractiveness levels $\vec{y}_t := (y_{1t}, \ldots, y_{NT})$. Letting M_t be the market size in period t, and p_i the unit contribution margin of category i, which can be assumed constant over time (Caro et al. 2014, Lago et al. 2013), we thus have that

$$R(\vec{y}_t) = M_t \left(\frac{\sum_{j=1}^{N} p_j y_{jt}}{y_{0t} + \sum_{j=1}^{N} y_{jt}} \right).$$

Since the retention of the attractiveness (or its decay) and the cost of increasing it can be different for each category, the retailer must balance the effort over time and across categories. We assume that retailer profits are discounted with rate $\beta \in [0, 1]$. With the objective to maximize the total discounted profit over T periods, the dynamic assortment planning problem can be written as follows:

$$\begin{aligned} \max_{\vec{a} \geq \vec{y}_t, \vec{u}_t \geq 0 \forall t \in \{1,..,T\}} \quad & \mathbb{E}\left[\sum_{t=1}^{T} \beta^{t-1} \left\{ R(\vec{y}_t) - C(\vec{u}_t) \right\} \right] \\ \text{s.t.} \quad & y_{it} = \varphi_i(u_{it}, \varepsilon_{it}, y_{it-1}) \ \forall i \in \{1, \ldots, N\} \ \forall t \in \{1, .., T\} \end{aligned} \tag{2}$$

Since the action $\vec{u}_t$ is taken with full knowledge of $\vec{x}_t := (x_{1t}, \ldots, x_{Nt})$, we can write this problem equivalently using a dynamic programming (DP) formulation, where $J_t(\vec{x}_t)$ is the profit-to-go from period t until T:

$$\begin{aligned} J_t(\vec{x}_t) = \quad & \max_{\vec{a} \geq \vec{y}_t \geq \vec{x}_t} \quad G_t = R(\vec{y}_t) - C(\vec{y}_t - \vec{x}_t) + \beta \mathbb{E}[J_{t+1}(\vec{x}_{t+1})] \\ & \text{s.t.} \quad x_{it+1} = \varepsilon_{it+1} y_{it} \qquad \forall i \in \{1, .., N\}. \end{aligned} \tag{3}$$

and $J_{T+1} \equiv 0$.

Note that we do not consider inventory constraints or costs. This is a general feature of most dynamic assortment models (e.g., Caro et al. 2014, Ulu et al. 2012). There are two reasons for this assumption. First, including inventory makes the analysis intractable and would force the consideration of stock-out based substitution, which is very hard to solve (Honhon et al. 2010); or of inventory influences on the attractiveness, such as the *broken assortment* effect identified in Smith and Achabal (1998) and Caro et al. (2010). In addition, if inventory had to be taken into account, decisions should be taken at the SKU level, and would need to incorporate inventory availability and progress of SKU attractiveness over time: this would make the state space very large. Second and most important, in most of the practical applications we have in mind, assortment decisions are taken before inventory is planned, and inventory can be adjusted to the assortment requirements: retailers first focus on deciding what efforts they should devote to each category, then make the decisions regarding the depth of the assortment, i.e., how many different SKUs to carry in every category, and then the inventory decisions, possibly including resupply, are taken. This is true for fast fashion players like Zara for instance, which are capable of providing more supply with a lead-time of 3-4 weeks. And these are precisely the firms capable of running dynamic assortments with closed-loop controls, which is our focus.

3.2 General properties

The DP in Equation (3) has a tractable structure.

Proposition 1. *When R is jointly concave in $\vec{y}_t$ and C is jointly convex in $\vec{u}_t$, then J_t is increasing and jointly concave in $\vec{x}_t$.*

Hence, one of the key requirements to obtain structure in our problem is that R is jointly concave in $\vec{y}_t$. This is the case when categories have identical margins, i.e., $p_1 = \ldots = p_N$. Furthermore, note that this result makes use of the joint concavity of $R(\vec{y}_t) - C(\vec{y}_t - \vec{x}_t)$ with respect to $(\vec{u}_t, \vec{x}_t)$, and requires that $\vec{u}_t$ can be chosen in a convex set. As a result, if u_{it} can only take integer values, the result does not hold, and other arguments must be used, e.g., supermodularity (Topkis 1998).

In order to obtain stronger results, in the remainder of the analysis we make two strong assumptions. First, we consider a stationary situation with a normalized market size $M_t = 1$, an outside option $y_{0t} = 1$ (these are scaling parameters that do not affect the results) and a decay ε_i with a time-independent distribution. Such assumption is relatively realistic in certain categories such as T-shirts, where sales and price points do not vary much over time. As can be observed from the data used in Lago et al. (2013), for a fast fashion company seasonality or price fluctuations are small compared to a traditional retailer where seasons are more apparent and markdowns are much more relevant. Second, we assume that the cost function is linear, i.e., $C(\vec{u}_t) = \sum_{j=1}^{N} c_j u_{jt}$. This is reasonable when u_{it} represents the number of new products designed by the company, for which a per-unit design cost, stable over time, is incurred. This linearity will allow us to decompose G_t as a sum of a nonlinear function of $\vec{y}_t$, plus a linear function of $\vec{x}_t$, which will provide interesting policy structures. Furthermore, to avoid trivial solutions where it is optimal not to carry product i, we assume that $p_i \geq c_i$ for $i \in \{1, \ldots, N\}$.

In addition, we initially assume, for part of our analysis, that all categories have identical margins equal to $p_i = p$, $i \in \{1, \ldots, N\}$. We then discuss the unequal margin case (in Sections 5.2 and 6.2). Equal margins is a reasonable assumption when the retailer offers similar price points in all categories. However, in most situations, this may not be satisfied, in which case the optimal policy may be extremely complex (Proposition 1 will not apply).

Under these assumptions, we can rewrite J_t as follows:

$$
\begin{aligned}
J_t(\vec{x}_t) = \quad & \max_{\vec{a}\geq\vec{y}_t\geq\vec{x}_t} \quad G_t = \sum_{i=1}^{N} c_i x_{it} + \sum_{i=1}^{N}\left(\frac{p_i y_{it}}{1+\sum_{j=1}^{N} y_{it}} - c_i y_{it}\right) + \beta\mathbb{E}[J_{t+1}(\vec{x}_{t+1})]. \\
& \text{s.t.} \quad x_{it+1} = \varepsilon_{it+1} y_{it} \qquad \forall i \in \{1,..,N\}.
\end{aligned} \tag{4}
$$

Given this structure, we first analyze the single-category model in Section 4. Then we study the multi-category problem, first relaxing the capacity constraint on the attractiveness, in Section 5, then with the added capacity constraint, in Section 6.

4 Single-Category Dynamic Assortment

4.1 Optimal policy

When the costs are linear we show that the optimal policy for a single category problem is one where at the beginning of every period the attractiveness of the category is raised up to a specific level if x_{it} is below this level and no action is taken otherwise. We call this an *assort-up-to* policy, since this is similar to the order-up-to level policies in inventory planning problems.

Theorem 1. *When $N = 1$, the optimal policy is an* assort-up-to *policy. In other words, there exists an assort-up-to level b_t such that $y_t^* = \max\{x_t, b_t\}$. Moreover, b_t is non-increasing in time: $b_t \geq b_{t+1}$.*

The theorem provides a quite simple characterization of the optimal policy: in period t, the retailer should examine how popular the category is, and make an effort only when it is lower than b_t. The structure directly follows from the linearity of cost, and as a result can be extended even when market size and outside option are non-stationary. When parameters are stationary, the optimal target levels b_t are decreasing over time. This means that the retailer will try to offer higher attractiveness earlier in the season, and reduce it as the number of periods remaining goes down. As an extreme case, imagine that there is no decay: then the attractiveness of the category will remain for T periods, and as a result, the higher T, the higher the willingness of the retailer to offer more choice.

We can further characterize the optimal assort-up-to levels, using the first-order condition:

$$
\frac{p}{(1+y_t)^2} - c + \beta\mathbb{E}\left[\varepsilon J'_{t+1}(\varepsilon y_t)\right] - \lambda_1 + \lambda_2 = 0, \tag{5}
$$

where λ_1 and λ_2 are the KKT multipliers associated with the constraints $a > y_t$ and $y_t > x_t$, respectively.

Proposition 2. *When $N = 1$ we define $\varphi^1 := \sqrt{\frac{p}{c}} - 1$ and $\varphi^\infty := \sqrt{\frac{p}{c(1-\beta\bar{\varepsilon})}} - 1$, with $\mathbb{E}[\varepsilon] = \bar{\varepsilon}$.*

- *for $T = 1$ (single period), $b_t = min(\varphi^1, a)$;*
- *for $T = \infty$ (infinite horizon), $b_t = min(\varphi^\infty, a)$.*

Interestingly, there are closed-form formulas for the extreme cases where $T = 1$ or $T = \infty$. For a single period, the investment in the category is independent of the decay, as there is a single opportunity to sell the items. For infinite horizons, the investment is increasing in β and $\bar{\varepsilon}$: in fact, the more "durable" a category is, the higher the incentive to put effort in it. All can be summarized in a normalized cost $c(1 - \beta\bar{\varepsilon})$.

Finally, when T is finite and larger than one, b_t will be recursively calculated, using Equation (5). The only case where this recursion can be solved in closed form is when ε is deterministic, equal to $\bar{\varepsilon}$.

Proposition 3. *When $N = 1$, and $\varepsilon = \bar{\varepsilon}$ with probability one for all $t \in \{1, \ldots, T\}$, define $\hat{b}_t$ as the unique b that satisfies:*

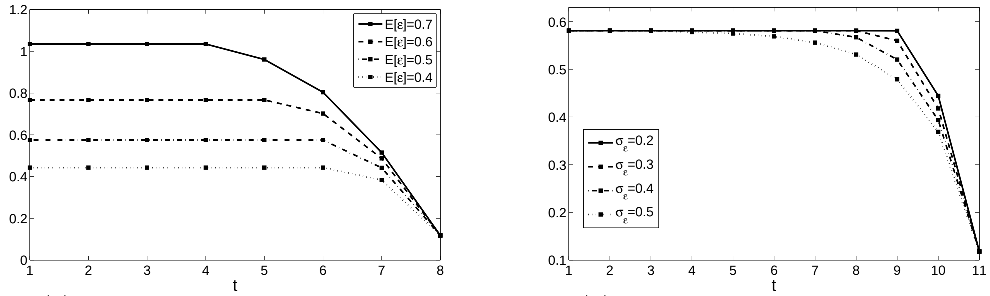
$$\sum_{\tau=0}^{T-t} \frac{(\beta\bar{\varepsilon})^\tau}{(1 + \bar{\varepsilon}^\tau b)^2} - \frac{c}{p} = 0. \tag{6}$$

Then:

- *$\hat{b}_t$ is a non-increasing sequence of t;*
- *there exists $t_{last} = \max\{t | \hat{b}_t \geq \varphi^\infty = \sqrt{\frac{p}{c(1-\beta\bar{\varepsilon})}} - 1\}$ such that;*
  for all $t \leq t_{last}$, $b_t = min(\varphi^\infty, a)$, and for $t > t_{last}$, $b_t = min(\hat{b}_t, a)$.

This result is similar to the result in Proposition 5 of Caro et al. (2014) in that the optimal attractiveness decreases in time. In Figures 1(a) and 1(b) the change of the assort-up-to levels in time can be seen as a function of the expectation of ε, $\bar{\varepsilon}$, and its standard deviation, σ_ε, respectively, for $c = 0.8$. For high $\bar{\varepsilon}$ φ^∞ is larger and b_t starts to decrease earlier toward the end of the selling horizon since attractiveness is large enough for the rest of the season and for high σ_ε b_t starts to decrease earlier in the season so as to avoid the risk of over-investing.

Note that all the results of this section can easily be extended to handle seasonality in the parameters. In that case, the assort-up-to levels need not be monotonic. We can numerically show that as the market size M_t changes throughout the planning season, the optimal assort



(a) b_t as function of $\bar{\varepsilon}$ for $\sigma_\varepsilon = 0.2$ (b) b_t as function of σ_ε for $\bar{\varepsilon} = 0.5$

Figure 1: The change of assort-up-to levels in time for $c = 0.8$ and $a = \infty$.

up-to levels typically increase or decrease with an increase or decrease of the market size in that period. A similar observation holds if the cost c_t changes throughout the planning season. When it comes to the outside option y_{0t}, its effect on the assort-up-to levels, whether they are going to increase or decrease, is not as obvious. For example, we can show that the stationary assort-up-to-level is a concave function of y_0, and hence an increase in y_{0t} may result in an increase or a decrease in the corresponding b_t.

4.2 Connection to inventory models and the effects of positive lead-times

The assortment planning problem discussed here is similar in some aspects to inventory problems with deteriorating inventory (Nahmias 1982) and inventory-dependent demand (Smith and Achabal 1998, Caro et al. 2010). Specifically, the attractiveness decay is analogous to the decay in the deteriorating inventory problems without demand. There, the natural quality of items such as fruit, vegetables or flowers decreases and products quickly lose their value (Blackburn and Scudder 2009). The fashion items that we consider have a similarly short market life cycle: after a period of popularity, they start losing their value due to the changes in consumer preferences. As a result, the actual attractiveness evolves as follows: $y_t = \varepsilon_t y_{t-1} + u_t$, where u_t is the amount of "fresh" products, while y_{t-1} is the amount of "old" products that already started decaying. Note that we are implicitly assuming that the decay among the "old" products is homogeneous.

Another interpretation of our model is to consider an inventory system with a stochastic demand that depends on the inventory level, as in Lago et al. (2013), $D_{t-1} = (1 - \varepsilon_t)y_{t-1}$. Then we have as the inventory balance equation $y_t = y_{t-1} - D_{t-1} + u_t$, which is identical to the attractiveness equation $y_t = x_t + u_t$.

Variations of our problem also share some similarities with the random yield problem. Indeed, in our model, the effort u_t comes into action immediately, in period t. However, in some cases the "new" products may experience some delay between the introduction decision and the actual appearance in the store, and as a result suffer some uncertainty in the attractiveness. If there is a positive lead-time for the effort to become effective, and decays of old and new items are different, the assortment problem is similar to the random yield problem. The state must be extended to keep track of the actions taken but not implemented yet. We can only solve it when the planned effort also decays in time together with the products already in the store. In any case, existence of a positive lead-time does decrease the benefit of dynamic assortment decisions. Large lead-times are actually the main reason for having to use open-loop policies instead of closed-loop feedback policies. We compare the two later, in Section 7.

5 Multi-Category Dynamic Assortment without Capacity Constraints

When there are multiple categories, the arguments used in Section 4 are no longer valid. Indeed, there are now several categories that evolve independently but interact in the revenue function. This coupling makes the optimal policy more complex.

5.1 Optimal policy for equal margins

We start discussing the case of equal margins across categories. This case makes the revenue function concave, and strong results can be derived.

Consider the first-order condition for the current profit G_t with respect to the attractiveness of category i:

$$0 = \frac{p}{(1 + \sum_{j=1}^{N} y_{jt})^2} + \beta \mathbb{E}\left[\varepsilon_{it+1} \frac{\partial J_{t+1}}{\partial x_{it+1}}(\vec{\varepsilon}_{t+1}\vec{y}_t)\right] - c_i + \lambda_{it} \tag{7}$$

where λ_{it} is the KKT multiplier associated with the constraint $y_{it} \geq x_{it}$. If these constraints are non-binding, then $\lambda_{it} = 0$ and then we can again characterize the optimal policy, this time for the multi-category problem.

Theorem 2. *There exist assort-up-to levels $b_{it} \geq 0$ such that, if for all i, $x_{it} \leq b_{it}$, then $y_{it}^* = b_{it}$; otherwise, y_{it}^* may depend on x_{it}.*

This result again stems from the linearity of the cost function, which makes profit a function of $\vec{y}_t$ and only linked to $\vec{x}_t$ through a constraint. As a result, if the constraints are non-binding, then the optimal policy is to use an assort-up-to policy, i.e., to raise attractiveness x_{it} to b_{it}. However, due to the coupling between categories, if a category is such that $x_{it} > b_{it}$, then, for $j \neq i$, even if $x_{jt} < b_{jt}$, it is possible that $y^*_{jt} \neq b_{jt}$. Generally, $y^*_{jt} < b_{jt}$, since the incentive to increase j's attractiveness is reduced because i's attractiveness is already too high. As in Section 4.2, we can link the assortment planning problem with multiple categories to a multi-product inventory problem. Specifically, in our problem costs are separable but products are coupled in the revenue function. In the inventory literature (Veinott 1965, Beyer et al. 2001 and Beyer et al. 2002 among others), products are usually coupled through the cost function or a storage capacity constraint. Despite these differences, we find similar structure in the optimal policy.

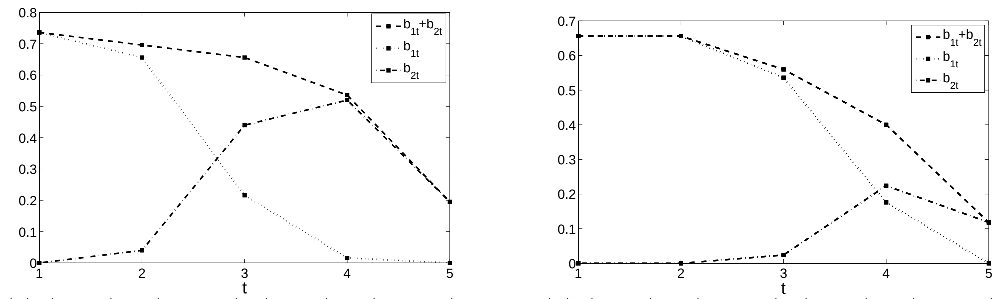
Interestingly, there are cases where some of the assort-up-to levels may be zero. In other words, it is best not to put any effort in some categories. This is for instance the case for $t = T$. Indeed, in the last period, Equation (7) with $\lambda_{it} = 0$ for all i implies that $\frac{p}{(1+\sum_{j=1}^{N} y_{jt})^2} = c_i$, which is only true when categories have the same per-unit effort cost. As a result, only investments in the category with the lowest cost should be considered in the last period. A similar result holds also for $T = \infty$.

Proposition 4. *For each $i \in \{1, \ldots, N\}$, let $\varphi_i^1 := \sqrt{\frac{p}{c_i}} - 1$ and $\varphi_i^\infty := \sqrt{\frac{p}{c_i(1-\beta\bar{\varepsilon}_i)}} - 1$. The assort-up-to levels satisfy:*

- *when $T = 1$ (single period), let i be one of the lowest-cost categories, i.e., $i = \arg\min_j c_j$: then $b_i = \varphi_i^1$, and $b_j = 0$ for $j \neq i$;*
- *when $T = \infty$ (infinite horizon), let i be such that $i = \arg\min_j c_j(1 - \beta\bar{\varepsilon}_j)$: then $b_{it} = \varphi_i^\infty$ and $b_{jt} = 0$ for $j \neq i$ for all t.*

Similar to Proposition 2, we are able to characterize the assort-up-to levels in the extreme cases $T = 1$ and $T = \infty$. Interestingly, the optimal solution with multiple categories is to invest in a single category in these cases. In general, however this is not true. Figure 2 illustrates how the assort-up-to levels may change over time. In this numerical example, the cost of investing in category 1 is larger ($c_1 = 0.8 > c_2 = 0.7$ for Figure 2(a) and $c_1 = 0.9 > c_2 = 0.8$ for Figure 2(b)) but its attractiveness lasts longer ($\bar{\varepsilon}_1 = 0.6 > \bar{\varepsilon}_2 = 0.5$ for Figure 2(a) and $\bar{\varepsilon}_1 = 0.6 > \bar{\varepsilon}_2 = 0.4$ for Figure 2(b)). We observe that the assort-up-to level is first large for category 1, and then it decreases as the season progresses, while there is no investment for category 2 at the beginning we observe positive investment later in the season. Thus, the investment is dependent on the remaining time to the end of the season: category 1 is more expensive to invest in, but its decay is smaller, hence it is better to invest in it only if there is enough time to recover the investment, and it is better to invest in category 2 later since it is cheaper. We can conclude that the profitability of the categories is not just measured by their costs but also by how they decay. Note that in Figure 2(a) investment in category 2 starts earlier since it is expected to decay less.

This behavior is reminiscent of Caro et al. (2014), where products with less decay should be launched earlier in the season (Proposition 4). However, our model does not have a fixed set of products to be introduced, but a cost for investing in a category. This makes the dynamics of our model different than Caro et al. (2014), where the introduction of products depends only on the decay: here, the timing of the investment is also crucial and depends both on the decay and the cost.



(a) $(c_1, c_2) = (0.8, 0.7)$, $(\bar{\varepsilon}_1, \bar{\varepsilon}_2) = (0.6, 0.5)$. (b) $(c_1, c_2) = (0.9, 0.8)$, $(\bar{\varepsilon}_1, \bar{\varepsilon}_2) = (0.6, 0.4)$.

Figure 2: Assort-up-to levels b_{it} over time, for $T = 5$, $(\sigma_{\varepsilon_1}, \sigma_{\varepsilon_2}) = (0.2, 0.2)$.

The dynamics of the optimal policy are thus complex. One may wonder whether there are non-extreme cases (e.g., $1 < T < \infty$) where some stronger results can be obtained. When decays are deterministic and identical across categories, we can show that the category with the lowest effort cost will be chosen in every period.

Proposition 5. *When $\varepsilon_i = \bar{\varepsilon}$ with probability one and common across categories, the optimal policy is an assort-up-to policy where investments are made only for the category with lowest per-unit effort cost c_i, i.e., $b_{it} = 0$ if $i \neq \arg\min_j c_j$.*

However, when categories have different decays, the assort-up-to levels may be difficult to obtain, even in the two-period case or the constant decay case. But we are able to characterize the solution for a two-period two-category problem with constant parameters $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$. In the following we discuss the solution for this special case.

Proposition 6. *Consider the case $T = 2$ and $N = 2$, with $p = 1$, and sort the categories by decreasing cost, i.e., $c_1 \geq c_2$. Then for $t = 2$, $b_{2t} = b_2^{single}$ and $b_{1t} = 0$. For $\varepsilon_i = \bar{\varepsilon}_i$ with probability one, if $\bar{\varepsilon}_1 \leq \bar{\varepsilon}_2$ then in $t = 1$ an investment is made only to the cheaper category 2. Moreover if $\bar{\varepsilon}_1 > \bar{\varepsilon}_2$ it is possible to invest in both if $c_1/\bar{\varepsilon}_1 < c_2/\bar{\varepsilon}_2$, in which case*

$$b_{1t} = \frac{\theta_1 - \theta_2\bar{\varepsilon}_2 + (\bar{\varepsilon}_2 - 1)}{\bar{\varepsilon}_1 - \bar{\varepsilon}_2}$$
$$b_{2t} = \frac{\theta_2\bar{\varepsilon}_1 - \theta_1 + (1 - \bar{\varepsilon}_1)}{\bar{\varepsilon}_1 - \bar{\varepsilon}_2}$$

with $\theta_1 = \sqrt{\frac{\bar{\varepsilon}_1 - \bar{\varepsilon}_2}{c_1 - c_2}}$ and $\theta_2 = \sqrt{\frac{1}{c_1 - \bar{\varepsilon}_1/\theta_1^2}}$.

A numerical example for this is when $c_1 = 0.5$, $c_2 = 0.427$, $\bar{\varepsilon}_1 = 0.7$, $\bar{\varepsilon}_2 = 0.5$, in which case $b_{11} = 0.73$ and $b_{21} = 0.29$. Note that investment in the more expensive category is worthwhile since its decay is smaller. And b_{1t} (b_{2t}) increases (decreases) as $\bar{\varepsilon}_1$ increases or c_1 decreases.

In contrast to such predictable decays, when decay is random, an investment can also be made simultaneously into different categories, as long as the categories are not too distinct from each other. Indeed, it is beneficial to diversify the investments so as to hedge against decay uncertainty. We show in Proposition 7 that, even though costs and decay distributions may be the same for all categories, the optimal investment should be to equally distribute the investments among the categories.

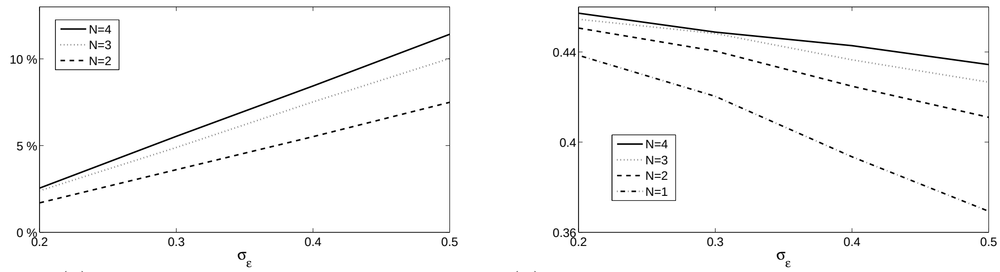
Proposition 7. *In the case where ε_{it}s are identically distributed and $c_i = c$ for all categories, assort-up-to levels are equal, $b_{it} = b_t$ for all i.*

To conclude this section, we investigate the role of category diversification (N) in the optimal policy. For this purpose, we compare numerically a scenario with N identical categories (same costs and evolution of the attractiveness), with a scenario with one category. We define the *Value of Hedging*, VoH, as the percentage gain in profits that is obtained by carrying N identical categories instead of one, as follows:

$$VoH = \frac{J_1^N - J_1^1}{J_1^1},$$

where J_1^N is the total expected profit for a system that carries N identical products.

In Figure 3(a), VoH is given for $T = 2$, $c = 0.8$ and different values of the spread of ε, from 0.2 to 0.5, when the attractiveness of all the products considered evolve independently according to the same decay distribution for $\bar{\varepsilon} = 0.5$. We observe that, as ε becomes more widely spread, even though the different categories evolve in the same manner, carrying more than one category is more profitable from a hedging perspective and thus VoH increases (Figure 3(a)). In Figure 3(b), the total assort-up-to level Nb_t is depicted for $t = 1$: it increases as N increases and decreases as σ_ε increases. This suggests that there is significant value from diversification and uncertainty hedging, especially when decays are highly uncertain. Finally, it is worth noting that these results are highly dependent on T. Indeed, this effect is quite significant for short horizons, but as the horizon grows longer, the differences are reduced, and in the limit $T = \infty$, VoH becomes zero (Proposition 4).



(a) VoH for varying values of σ_ε

(b) Total assort-up-to level for varying values of σ_ε

Figure 3: VoH and total assort-up-to level Nb_1 for $T = 2$, $c = 0.8$, $\bar{\varepsilon} = 0.5$.

5.2 Unequal margins

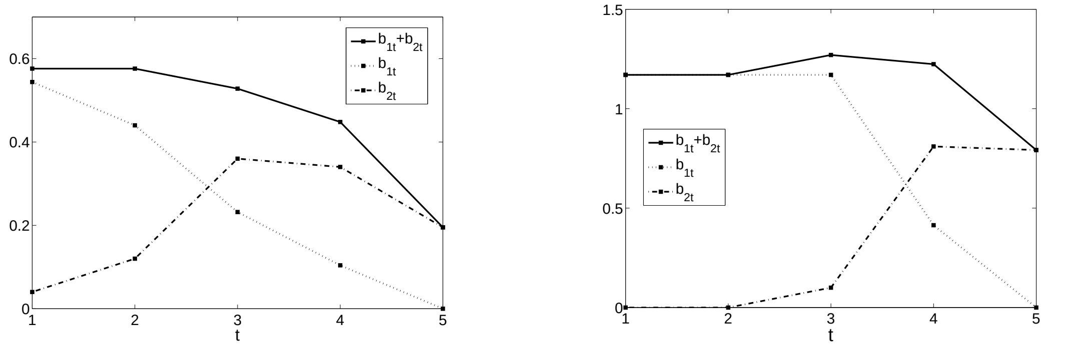
As mentioned earlier, the assortment planning problem is not concave when the margins for different categories are different, i.e., $p_i \neq p_j$ for some i, j. Hence, it is difficult to solve this problem, and Proposition 1 does not apply. In fact, unequal margin assortment planning is a notoriously difficult problem well studied in the literature. In a static setting, Talluri and van Ryzin (2004) show that a revenue-order assortment is optimal, one where only the highest margin products are selected; Rusmevichientong et al. (2010) incorporate a capacity constraint. Unfortunately, in dynamic environments, the problem is not tractable, see Caro et al. (2014). Our model faces similar tractability issues. Fortunately, we consider continuous decision variables y_{it}, and, although general fractional programs are difficult to solve (Schaible and Shi 2003), in some circumstances some results can be derived. Specifically, for the extreme cases where $T = 1$ or $T = \infty$ (starting with zero attractiveness), an assort-up-to level policy is optimal.

Proposition 8. *For each $i \in \{1, \ldots, N\}$, let $\varphi_i^1 := \sqrt{\frac{p_i}{c_i}} - 1$ and $\varphi_i^\infty := \sqrt{\frac{p_i}{c_i(1-\beta\bar{\varepsilon}_i)}} - 1$, and assume $x_{i1} = 0$. Then, investing in only one category always results in the highest expected profit* *for the single-period and infinite-horizon problems. The optimal policy is an assort-up-to policy as in Theorem 2, with assort-up-to levels defined as follows:*

- *when* $T = 1$ *(single period), let* i *be one of the lowest-cost categories, i.e.,* $i = \arg\max_j \sqrt{p_j} - \sqrt{c_j}$*: then* $b_i = \varphi_i^1$*, and* $b_j = 0$ *for* $j \neq i$*;*
- *when* $T = \infty$ *(infinite horizon), let* i *be such that* $i = \arg\max_j \sqrt{p_j} - \sqrt{c_j(1 - \beta\bar{\varepsilon}_j)}$*: then* $b_{it} = \varphi_i^\infty$ *and* $b_{jt} = 0$ *for* $j \neq i$ *for all* t*.*

This result provides a rich characterization of the optimal policy in the extreme cases $T = 1$ and $T = \infty$. We show that concentrating in one category is optimal: it is the one where $\sqrt{p_j} - \sqrt{c_j}$ (single period) or $\sqrt{p_j} - \sqrt{c_j(1 - \beta\bar{\varepsilon}_j)}$ (infinite horizon) is largest. For instance, when costs and decays are identical, choose the highest margin category; when margins are identical, choose the category with lowest normalized cost, either c_j or $c_j(1 - \beta\bar{\varepsilon}_j)$ depending on the horizon, as in Proposition 4.

Again, it is difficult to characterize the optimal solution in general, and in particular an assort-up-to policy may not be optimal. We can nevertheless compute the optimal assort-up-to level $b_{it} := y^*_{it}(0)$ if the starting attractiveness x_{it} is zero for all i: when $x_{it} \leq b_{it}$ for all i, then $y^*_{it}(\vec{x}_t) = b_{it}$. These assort-up-to levels deserve some study. For a two-category problem, let us assume that $c_1 \geq c_2$ and $p_1 \geq p_2$. Then the optimal decision could be to invest in any of the two products in any given period, i.e., $b_{it} > 0$ for all i. In Figure 4(a) we show an example for $T = 5$, $(p_1, p_2) = (1, 0.9)$, $(c_1, c_2) = (0.8, 0.7)$, $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0.5$, $\sigma_{\varepsilon_1} = 0.3$ and $\sigma_{\varepsilon_2} = 0.2$. At the beginning of the season an investment is made only to the high price/high cost category, when the more expensive investment can be recovered, then investment is made to the low price/low cost one towards the end. This switch is similar to the switch in the equal margins case where it was due to the differences in the expected decays. Moreover, in contrast with the equal margins case, the sum of the assort-up-to levels $\sum_{i=1}^N b_{it}$ can also be non-monotonic in time. We give an example for this in Figure 4(b) (for $T = 5$, $(p_1, p_2) = (1, 0.8)$, $(c_1, c_2) = (0.4, 0.25)$, $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0.5$, $\sigma_{\varepsilon_1} = 0.3$ and $\sigma_{\varepsilon_2} = 0.4$). In general, the decisions in the unequal margin case more complicated by the relative differences of price, cost and decays.



(a) b_{it}s for $(p_1, p_2) = (1, 0.9)$, $(c_1, c_2) = (0.8, 0.7)$, $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0.5$, $(\sigma_{\varepsilon_1}, \sigma_{\varepsilon_2}) = (0.3, 0.2)$.

(b) b_{it}s for $(p_1, p_2) = (1, 0.8)$, $(c_1, c_2) = (0.4, 0.25)$, $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0.5$, $(\sigma_{\varepsilon_1}, \sigma_{\varepsilon_2}) = (0.3, 0.4)$.

Figure 4: Assort-up-to levels b_{it} for unequal margins, for $T = 5$.

6 Multi-Category Dynamic Assortment Planning Model with Capacity Constraints

In this section we consider the case when there is a stationary capacity constraint on the attractiveness of categories, $\vec{a} \geq \vec{y_t}$. For the capacitated problem the results of Propositions 4 and 8 do not hold anymore: we show that for the special cases of $T = 1$ and $T = \infty$, both for equal and unequal margin problems, the retailer might need to diversify the assortment instead of carrying only one category.

6.1 Equal margins

If margins are equal, the result of Theorem 2 can be easily extended to include the added capacity constraint.

Proposition 9. *When the attractiveness of category i is limited to at most a_i, the assort-up-to levels for $T = 1$ and $T = \infty$ are found by using the following algorithm, which is polynomial in the number of categories N. For $T = 1$:*

1. *Sort categories in an increasing order of costs, i.e., such that $c_{i_1} \leq c_{i_2} \leq \cdots \leq c_{i_N}$. For each $i \in \{1, \ldots, N\}$, let $\varphi_i^1 := \sqrt{\frac{p}{c_i}} - 1$.*

2. *Start with the first category $k = 1$.*

3. *If $\varphi_{i_k}^1 - \sum_{j=1}^{k-1} a_{i_j} < a_{i_k}$,*

(a) then set $b_{i_k} = \max\left\{\varphi^1_{i_k} - \sum_{j=1}^{k-1} a_{i_j}, 0\right\}$ *and* $b_{i_k} = 0$ *for* $j > k$*;*

(b) else $b_{i_k} = a_{i_k}$ *and move to the next category* $k+1$*.*

For $T = \infty$*, using the same algorithm with* $\varphi^\infty_i := \sqrt{\frac{p}{c_i(1-\beta\bar{\varepsilon}_i)}} - 1$ *instead of* φ^1_i *yields the optimal solution.*

As we can see, the algorithm greedily adds attractiveness to each category until either the optimal amount is reached (φ^1_i or φ^∞_i) or the capacity constraint becomes binding. This implies that the firm will invest in more than one category: capacities will result in diversification.

Furthermore, we can also extend Proposition 5. If $\varepsilon_i = \bar{\varepsilon}$ for all categories we can show that an investment is made starting with the category with the smallest cost and categories are added in an increasing order of costs in any period. The optimal solution to this special case is actually similar to the one we show for the extreme cases $T = 1$ and $T = \infty$.

6.2 Unequal margins

Assuming again that $x_{i1} = 0$ for all $i \in \{1, \ldots, N\}$, the single period problem with unequal margins subject to capacity constraint can be written as follows.

$$\max_{z \geq 0} \left\{ \max_{\vec{b}} \sum_{j=1}^{N} b_j \left(\frac{p_j}{1+z} - c_j \right) \text{ s.t. } \vec{a} \geq \vec{b} \geq 0 \text{ and } \sum_{j=1}^{N} b_j = z \right\}$$

where z is the total attractiveness, i.e., $\sum_{j=1}^{N} b_j$. The inner maximization is a linear problem in $\vec{b}$ given z. Let $d_j(z) = \frac{p_j}{1+z} - c_j$.

Let $K(z)$ be the set of categories that satisfy $b_j = a_j$ at optimality (note that this set might be changing as z changes). Of course, the optimal solution depends on the ordering of $d_j(z)$: to maximize the profit, it is optimal to introduce the categories with the largest d_j and their target level should be as high as possible (equal to a_j). There is possibly one category i in the optimum with $b_i < a_i$, with smaller d_i than the ones in $K(z)$. Let $I(z) = \{i\}$.

Let $LB \leq z \leq UB$ be an interval for which the ordering of $d_j(z)$ is fixed and $\sum_{j \in K(z)} a_j \leq z \leq \sum_{j \in K(z)} a_j + a_i$. In this range of z, $K(z)$ and $I(z)$ do not change. Thus, the objective function can be written as follows:

$$\sum_{j \in K(z)} a_j d_j(z) + \left(z - \sum_{j \in K(z)} a_j \right) d_i(z)$$

Thus, to optimize over z in this range, we have the following first-order condition:

$$0 = d_i(z) - \frac{\sum_{j \in K(z)} a_j p_j + \left(z - \sum_{j \in K(z)} a_j\right) p_i}{(1+z)^2} = \frac{p_i - \sum_{j \in K(z)} a_j (p_j - p_i)}{(1+z)^2} - c_i$$

Thus, within the interval $z \in [LB, UB]$, the optimal z can be equal to the unique solution to the previous equation (if it exists), or one of the extremes LB or UB. In addition, the number of possible orderings of d_js is $\frac{N(N-1)}{2} + 1$. Taking also into account the extreme case of $I = \emptyset$ the maximum number of evaluations is $\frac{N}{2}(N^2 - N + 4)$. As a result, by evaluating a polynomial number of choices we can find the optimal solution, as formalized in the next result.

Proposition 10. *When the attractiveness of category j can be increased up to a_j the optimal solution can be found for $T = 1$ and $T = \infty$ by evaluating a limited number of profits, which is polynomial in the number of categories N. For $T = 1$:*

1. *For every i, j, determine z_{ij} such that $d_i(z) = d_j(z)$. Let $Z = \left\{z_{ij} | i, j \in \{1, \ldots, N\}\right\}$. Let $z_{(\ell)}$ the ℓ-lowest value in Z, $\ell \in \{1, \ldots, |Z|\}$.*

2. ***Upper loop**. For every $\ell \in \{1, \ldots, |Z| + 1\}$, let $(i_1, \ldots, i_N)$ the sequence in which $d_j(z)$ are sorted, i.e., $d_{i_1}(z) \geq \ldots \geq d_{i_N}(z)$ for $z \in [z_{(\ell-1)}, z_{(\ell)}]$, where $z_{(0)} = 0$ and $z_{(|Z|+1)} = \infty$.*
   ***Lower loop**. For $(K, I) \in \left\{(\emptyset, \{i_1\}), (\{i_1\}, \{i_2\}), (\{i_1, i_2\}, \{i_3\}), \ldots, (\{i_1, \ldots, i_{N-1}\}, \{i_N\})\right\}$, let $LB_{\hat{z}} = \max\left\{z_{(\ell-1)}, \sum_{k \in K} a_k\right\}$ and $UB_{\hat{z}} = \min\left\{z_{(\ell)}, \sum_{k \in K \cup I} a_k\right\}$.*

   - *For $(K, I) = (\emptyset, \{i_1\})$ set $b_{i_1} = min(\varphi^1_{i_1}, z_\ell)$ and $b_j = 0$ for $j \neq i_1$. Evaluate the profit function.*
   - *For every other (K, I) calculate $e = -\sum_{k \in K} a_k(p_k - p_i) + p_i$. If $e \geq 0$, let $\hat{z} = \sqrt{\frac{e}{c_i}} - 1$. If $LB_{\hat{z}} \leq \hat{z} \leq UB_{\hat{z}}$, assign $b_k = a_k$ for $k \in K$, $b_i = \hat{z} - \sum_{k \in K} a_k$ and $b_j = 0$ for $j \notin K \cup I$. Evaluate the profit function.*
   - *Assign $b_k = a_k$ for $k \in K \cup I$ and $b_j = 0$ for $j \notin K \cup I$. Evaluate the profit function.*

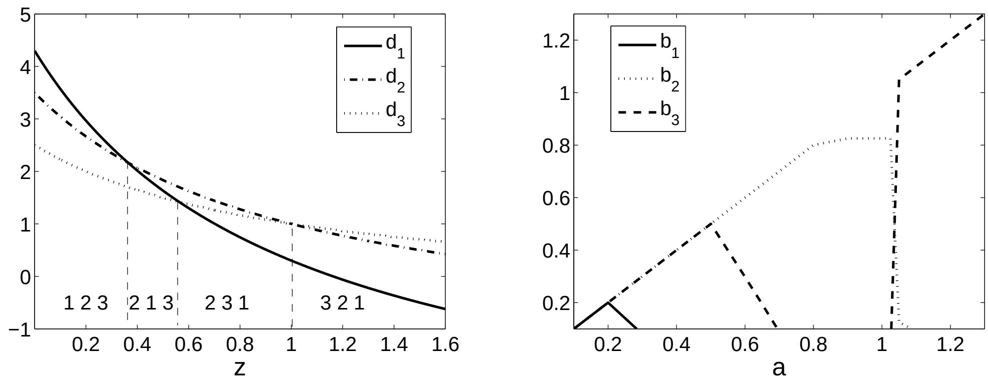
3. *Choose the ℓ, K, I that gives the best profit, which is the optimal solution.*

For $T = \infty$ the optimal solution is found by replacing c_i with $c_i(1 - \beta\bar{\varepsilon}_i)$ in the algorithm.

To illustrate how the algorithm works, consider an example with $\vec{p} = (8, 5, 3)$ and $\vec{c} = (3.7, 1.5, 0.5)$. If there were no constraints on the attractiveness we know from Proposition 8 that the optimal would be to invest in category 3 with $b_3 = \varphi^1_3 = 1.45$. With the constraints there might be a diversification. In Figure 5(a) the possible orderings for d_js can be observed:

for $0 \leq z \leq 0.36$ $d_1 \geq d_2 \geq d_3$; for $0.36 \leq z \leq 0.56$ $d_2 \geq d_1 \geq d_3$; for $0.56 \leq z \leq 1$ $d_2 \geq d_3 \geq d_1$; $1 \leq z$ $d_3 \geq d_2 \geq d_1$.

Consider for example the capacity vector $\vec{a} = (0.6, 0.6, 0.6)$. In the first interval, we evaluate $(K = \emptyset, I = \{1\})$; in the second interval we evaluate $(K = \emptyset, I = \{2\})$; in the third interval we evaluate $(K = \{2\}, I = \{3\})$ and $(K = 2, I = \emptyset)$; finally in the fourth interval, we evaluate $(K = \{3\}, I = \{2\})$ and $(K = \{2, 3\}, I = \emptyset)$. The best profit is equal to 1 with $b_3 = 0.6$ and $b_2 = 0.3$. With tighter capacities $\vec{a} = (0.3, 0.3, 0.3)$, the profit is reduced to 0.9 with $b_2 = b_3 = 0.3$ and $b_1 = 0.08$. In contrast, a more relaxed problem gives a profit of 1.5 for $\vec{a} = (1.3, 1.3, 1.3)$ with $b_3 = 1.3$. Figure 5(b) shows the categories chosen in the optimal solution and how the optimal attractiveness of different categories are distributed as a changes. While the optimal profits are obviously increasing in a, the optimal solution is non-monotonic, and profits are generally not convex nor concave.



(a) Value of (d_1, d_2, d_3) as a function of z and possible orderings.

(b) b_is as a function of $\vec{a} = (a, a, a)$.

Figure 5: Example for unequal margin algorithm with $\vec{p} = (8, 5, 3)$; $\vec{c} = (3.7, 1.5, 0.5)$.

7 The Value of Closed-Loop Optimization

The focus of this paper is on controlling assortments with the latest information. In this section, we are interested in quantifying the value of responsiveness, i.e., how much is gained by using a closed-loop approach, compared to dynamic open-loop and also to static assortment policies. For simplicity, we focus on the case without capacity constraints. The aim is to understand the importance of updating the decisions as the season progresses and to see if the flexibility of postponing decisions is worthwhile. We find that gains are significant, and more importantly, that it is closed-loop optimization that creates most of the value of dynamic assortments vs. static ones.

Under an open-loop policy, decisions are taken irrespective of unplanned changes in the system state. In other words, the decisions u_{it} are taken at the beginning of the planning horizon, i.e., at $t = 1$, as in Caro et al. (2014). In the open-loop case, the assortment planning problem for the multi-category starting at $\vec{x}_1 = 0$ is given as follows:

$$\pi_{open} = \max_{\vec{u}_1,\ldots,\vec{u}_T \geq 0} \mathbb{E}\left[\sum_{t=1}^{T} \beta^{t-1} \sum_{i=1}^{N} \left\{ \frac{p_i \sum_{\tau=1}^{t} \left(\prod_{s=\tau+1}^{t} \varepsilon_{is}\right) u_{i\tau}}{1 + \sum_{j=1}^{N} \sum_{\tau=1}^{t} \left(\prod_{s=\tau+1}^{t} \varepsilon_{js}\right) u_{j\tau}} - c_{it} u_{it} \right\}\right] \quad (8)$$

where in the numerator and denominator $\sum_{\tau=1}^{t} \left(\prod_{s=\tau+1}^{t} \varepsilon_{is} \right) u_{i\tau} = y_{it}$. Note that here the expectation is taken over all ε_{it}, and all the decisions are given before the selling season starts. The objective function reflects how every period the attractiveness is increased by u_{it} and how it decays with parameter ε_{it}. This problem is again concave under equal margins. Since we cannot generally find a closed-form solution, we find the optimal solution numerically. Note that this is a complex task because, even when decays follow a two-point distribution, the objective is a sum of $O(2^{NT})$ ratios of linear functions of u_{it}, which makes it computationally intractable. We can use optimization methods coupled with Montecarlo simulation, but we prefer to obtain exact solutions so we keep T small. In contrast, a closed-loop is computationally simpler, as we "only" need to run the dynamic programming algorithm with N dimensions.

We define the *Value of Responsiveness*, VoR, as the percentage gain in expected profit of the closed-loop optimal policy over the open-loop optimal policy:

$$VoR = \frac{\pi_{closed} - \pi_{open}}{\pi_{open}},$$

where π_{closed} is the expected profit of the system under the closed-loop policy, and π_{open} is defined in Equation (8).

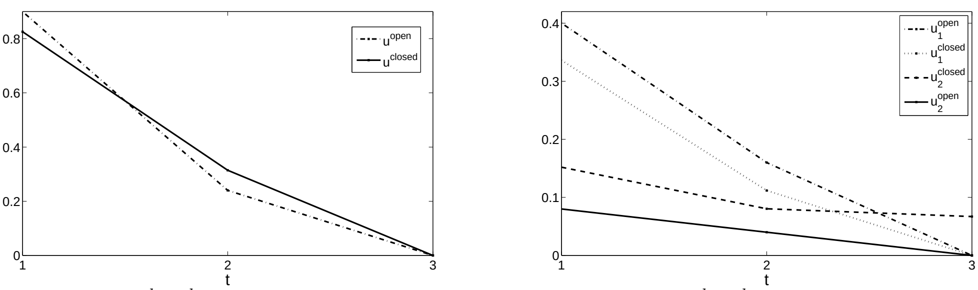
Furthermore, we compare these dynamic solutions to front-loaded static policies, those that only introduce changes in the first period. In other words, the best static policy solves the program in (8) with the additional constraint $\vec{u}_2 = \ldots = \vec{u}_T = 0$. We can thus define the *Value of Novelty*, VoN, as the percentage gain in expected profit of the open-loop optimal policy over the best static policy:

$$VoN = \frac{\pi_{open} - \pi_{static}}{\pi_{static}},$$

where π_{static} is the expected profit of the system under the best static policy.

When there is more than one category in the assortment the investment to be made depends on the relative risk and gain from these different categories. Here we do not consider trivial cases where investing in one of the products is far more profitable than investing in others, such as a category with much lower expected decay and much higher margin. In general, the behavior of assort-up-to levels are similar for both open-loop and closed-loop policies, but the distribution of the investments across time for the two policies are different, which results in different profits.

In Figure 6, we depict the expected investment made in each category to see the difference between two policies for $N = 1$ and $N = 2$, and $T = 3$. For $N = 1$ (with $p = 1$, $c = 0.9$, $\bar{\varepsilon} = 0.5$, $\sigma_\varepsilon = 0.1$) the investment in the open-loop policy is smaller than in the closed-loop policy for $t = 1$ and it is the opposite for $t = 2$ and there is no investment in the last period, while there is a small investment for the closed-loop policy. This distribution of investment results in a difference of 9.8% in profits. This increase in profits is mainly due to the better timing of investments under closed-loop. For $N = 2$, we compare u_{it}^{open} and u_{it}^{closed} for the following parameters $(p_1, p_2) = (1, 0.9)$, $(c_1, c_2) = (0.8, 0.75)$, $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0.5$, $(\sigma_{\varepsilon 1}, \sigma_{\varepsilon 2}) = (0.5, 0.5)$. In the open-loop policy in $t = 1, 2$ more investment is made in category 1, which has the higher margin and is more profitable in expectation, and no investment is made at the end of the season, while with the closed-loop policy an investment is made until the end of the season as the decay is observed. With this distribution of investment the expected profit for closed-loop is 5.7% better.

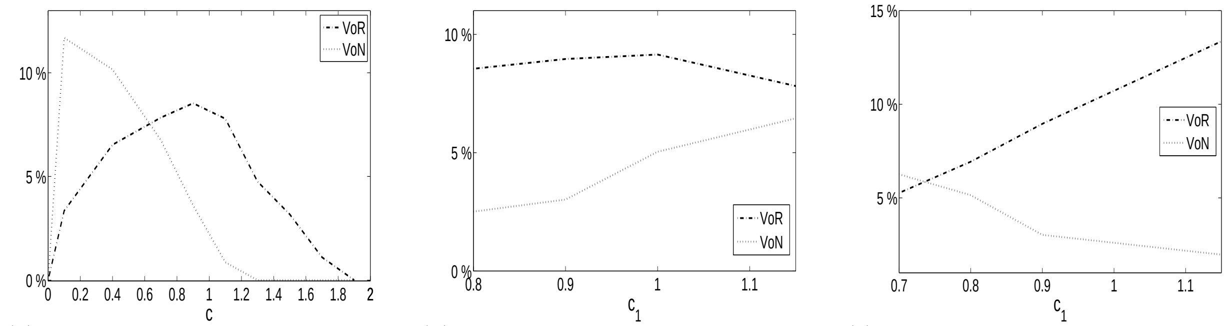


(a) u_t^{open} and $\mathbb{E}u_t^{closed}$ for $N = 1$, $p = 1$, $c = 0.6$, $\bar{\varepsilon} = 0.5$, $\sigma_\varepsilon = 0.1$

(b) u_{it}^{open} and $\mathbb{E}u_{it}^{closed}$ for $N = 2$, $(p_1, p_2) = (1, 0.9)$ and $(c_1, c_2) = (0.8, 0.75)$ and $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0.5$, $(\sigma_{\varepsilon_1}, \sigma_{\varepsilon_2}) = (0.5, 0.5)$

Figure 6: Expected investment as a function of time for $N = 1$ and $N = 2$, $T = 3$.

Figures 7, 8 and 9 illustrate the relative performance of the three policies (closed-, open-loop and static), i.e., the value of responsiveness VoR and of novelty VoN, as a function of the parameters. We do not display the actual profits, which, as one would expect, go down as cost increases, expected decay is stronger or decay becomes more variable: the system becomes either more expensive to maintain or less durable.



(a) VoR and VoN for $N = 1$, $p = 1$.

(b) VoR and VoN for $N = 2$ with fixed $c_2 = 0.7$ and $(p_1, p_2) = (1.2, 1)$.

(c) VoR and VoN for $N = 2$ with $c_2 = c_1 - 0.2$ and $(p_1, p_2) = (1.2, 1)$.

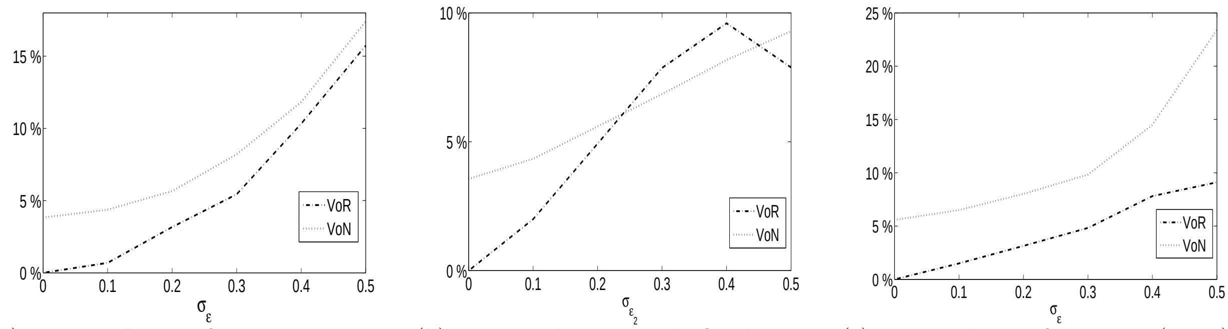
Figure 7: Comparison of VoR and VoN for one or two categories, as a function of cost for $\bar{\varepsilon} = 0.55$ and $\sigma_\varepsilon = 0.35$.

In Figure 7(a) the change in VoR and VoN is given for c from 0 to 2 for $p = 1$. For $N = 2$, in Figure 7(b) VoR and VoN are given for c_1 from 0.8 to 1.15 for fixed $c_2 = 0.7$ and $(p_1, p_2) = (1.2, 1)$, while in Figure 7(c) both costs change so as to keep the same margin, i.e., p_1, p_2, c_1 same as in Figure 7(b) and $c_2 = c_1 - 0.2$. $\bar{\varepsilon} = 0.55$ and $\sigma_\varepsilon = 0.35$ is common to all categories considered here.

For $N = 1$ both VoR and VoN increase and then decrease in cost. When cost is zero, any policy will put infinite effort in the assortment, thereby capturing 100% of sales. As cost goes to infinity, no investment will be made and profits will be zero. Thus, at the extremes, $VoR = VoN = 0$. In the middle range, when it is rather inexpensive to improve the assortment, VoR and VoN increase in cost. This is because, when cost goes up, the open-loop policy will have to reduce the investment, while the closed-loop one is still capable of putting up the investment only when it is needed. This advantage becomes more significant as cost becomes higher. Similarly, the static policy suffers more from the cost increase because all investment must be done upfront and thus later periods' attractiveness will be reduced significantly, while this is not so marked under open-loop policies. As cost becomes much higher, the reverse situation occurs: investments under closed- and open-loop policies are severely reduced, thereby reducing VoR. VoN also decreases dramatically as cost increases, because when investment is more expensive, later investments are avoided in open-loop policies (as they have less time to pay back), and hence the optimal open-loop policy resembles the static one more and more.

For the two-category case $N = 2$, VoR depends highly on the relative values of the costs and margins. In Figure 7(b) where c_2 is fixed, as c_1 increases VoR first increases, because, by using the feedback on actual decays, a better distribution of the investment between the two categories is possible. But when c_1 is too high, then it is more profitable to invest in category 2 only, for both policies, and the observation of the realized decay is not as valuable anymore (note that when this happens we are in the single-category case, in fact, we observe the same VoR and VoN for the single-category case with $c_2 = 0.7$). After this point, if c_1 was increased further, VoR and VoN would remain the same.

When both costs increase together, then there is no switch between categories as we discussed before, see Figure 7(c). The dynamics are similar to Figure 7(a), and VoR and VoN are generally increasing and then decreasing. In the range of the figure, when both costs are high, novelty is avoided by the open-loop policy, because it is harder to recover the costs incurred. Hence, the option to introduce new products during the season is not as advantageous, and VoN decreases. Similarly, as both costs increase, it is more critical to use the latest information to decide which category to invest in and how much, and VoR increases with costs.



(a) VoR and VoN for $N = 1$, $p = 1$, $c = 0.75$.

(b) VoR and VoN with fixed $\sigma_{\varepsilon_1} = 0.4$, for $N = 2$, $(c_1, c_2) = (0.8, 0.75)$, $(p_1, p_2) = (1, 1)$.

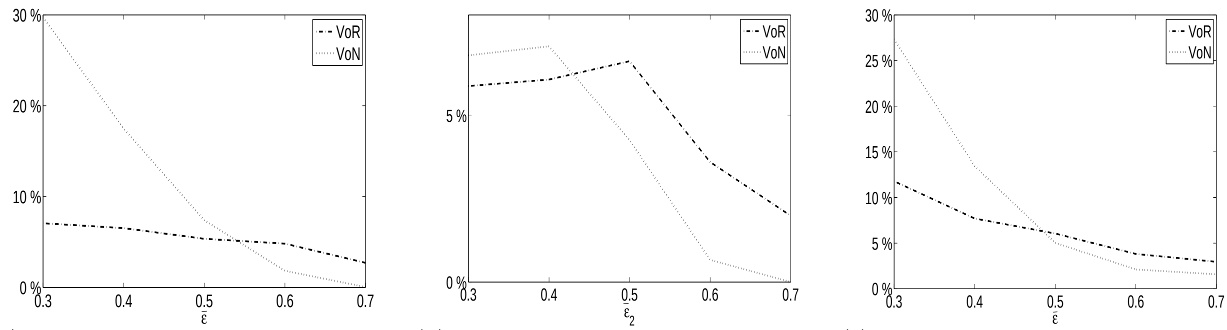
(c) VoR and VoN for $N = 2$, $(c_1, c_2) = (0.4, 0.25)$, $(p_1, p_2) = (1, 0.8)$.

Figure 8: Comparison of VoR and VoN for one or two categories, as a function of standard deviation of ε for $\bar{\varepsilon} = 0.5$.

In Figure 8, we consider decay distributions with varying standard deviations from 0 to 0.5 with $\bar{\varepsilon} = 0.5$. For $N = 1$, the change in VoR and VoN are given in Figure 8(a), with $p = 1$ and $c = 0.75$. Figure 8(b) is for $N = 2$, $(p_1, p_2) = (1, 1)$, $(c_1, c_2) = (0.8, 0.75)$ and only the decay uncertainty of the second category, σ_{ε_2}, is varied. In Figure 8(c) decay distributions are the same for the two categories, $(p_1, p_2) = (1, 0.8)$ and $(c_1, c_2) = (0.4, 0.25)$.

We see that for $N = 1$ and for $N = 2$ with equal uncertainty (Figures 8(a) and 8(c)), VoR is increasing as standard deviation of the decay increases. Indeed, closed-loop controls allow the retailer to exert efforts only when they are needed, while open-loop decisions commit to them regardless of the evolution of attractiveness, which is increasingly costly as uncertainty increases: VoR is higher. VoN also increases in the uncertainty level. This is because the ability to introduce products later in the season makes them more cost-efficient, as they are safe from possible decays earlier in the season. Recall that investing in category i an amount of u_{i1} only results in $\varepsilon_{i2}\varepsilon_{i3}u_{i1}$ in period 3, so this is more expensive than investing u_{i3}. Thus, the higher the variance, the higher this advantage of open-loop vs. static policies. Because of this inefficiency, even when $\sigma = 0$, VoN is positive.

In addition, when there are two categories with different variance parameters, high VoR is obtained as long as the two categories are similar in variance, but as they become more distinct there may be a switch to the category that is expected to be more profitable, and the value of the feedback is not as important anymore. In Figure 8(b), VoR decreases when uncertainty is high, as there is a switch of the investment to the lowest cost category (to decrease the risk of not recovering the investment). We can conclude that VoR is, in general, a function of the total uncertainty but also of the difference between the categories.



(a) VoR and VoN for $N = 1$, $\sigma_\varepsilon = 0.3$, $p = 1$, $c = 0.8$.

(b) VoR and VoN with fixed $\bar{\varepsilon}_1 = 0.5$, for $N = 2$, $\sigma_{\varepsilon_1} = 0.3$, $\sigma_{\varepsilon_2} = 0.2$, $(c_1, c_2) = (0.8, 0.7)$, $(p_1, p_2) = (1, 0.9)$.

(c) VoR and VoN for $N = 2$, $\sigma_{\varepsilon_1} = \sigma_{\varepsilon_2} = 0.2$, $(c_1, c_2) = (0.8, 0.8)$, $(p_1, p_2) = (1, 1)$.

Figure 9: Comparison of VoR and VoN for one or two categories, as a function of expected value of ε.

Figure 9(a) depicts the change in VoR and VoN with varying $\bar{\varepsilon}$ from 0.4 to 0.7, for $N = 1$, $\sigma_\varepsilon = 0.3$, $p = 1$ and $c = 0.8$. For $N = 2$, in Figure 9(b) VoR and VoN are given for fixed $\bar{\varepsilon}_1 = 0.5$,

$\sigma_{\varepsilon_1} = 0.3$ and varying $\bar{\varepsilon}_2$ from 0.3 to 0.7 for $(c_1, c_2) = (0.8, 0.7)$ and $(p_1, p_2) = (1, 0.9)$; in Figure 9(c) the results are for the same distribution for decays, with $\sigma_{\varepsilon_1} = \sigma_{\varepsilon_2} = 0.2$, $(c_1, c_2) = (0.8, 0.8)$ and $(p_1, p_2) = (1, 1)$.

The impact of the average decay $\bar{\varepsilon}$ is similar to that of the standard deviation: if products that decay faster, i.e., if $\bar{\varepsilon}$ is smaller, VoR and VoN will be higher for $N = 1$ and $N = 2$ with categories sharing the decay distribution. Note that when $\bar{\varepsilon} = 1$, which means $\varepsilon = 1$ with probability one, everything will be offered at the beginning of the planning season since there will be no decay, hence there will not be any difference between different policies. We again observe a non-monotonic behavior of VoR and also for VoN for the 2-category case when $\bar{\varepsilon}_1$ is fixed. Indeed, as $\bar{\varepsilon}_2$ increases, both first increase since the two categories are not too different. As $\bar{\varepsilon}_2$ increases further, then the retailer switches the investment to only category 2 with lower decay and lower cost, and the differences in policies are lower in that case as observed in Figure 9(b).

In summary, the value of dynamic assortments comes from two levers. First, there is value in the ability of introducing novelty (e.g., new products) during the season, as opposed to launching an assortment in the first period and watching it decay; this is measured by VoN. Second, there is also value in the ability of using the latest information to update dynamically the planned assortment; this is measured by VoR. Our numerical study reveals that in most scenarios, the second lever (VoR) is quite significant. There are regimes where the entire value is derived from this, e.g., costs above 1.4 in Figure 7(a). This implies that closed-loop decision-making may be the key operational lever in setting up dynamic assortments.

8 Conclusions and Further Research

In this paper we study the dynamic assortment planning problem, typical in fashion apparel retailing. Assuming a fixed set of categories, we analyze how to maximize profits by adjusting the attractiveness levels of its products during the selling season. When the ordering cost is linear, we characterize the optimal closed-loop policy for single and multiple category problems. We find that an assort-up-to policy is optimal for a single-category problem. For the multi-category problem with the equal-margin case we show that as long as the attractiveness of categories are all below their respective assort-up-to levels the optimal policy is to invest up-to these levels, otherwise the optimal solution may depend on the starting attractiveness levels. For the special cases of planning for very short and very long terms we show that it is best to choose only one category for investment, as long as there is no constraint on the maximum achievable attractiveness for any category. However, if the attractiveness of any category is limited then there may be need for diversification. We find closed-form solutions for the uncapacitated problems and we define algorithms to find the solutions for the capacitated ones. Finally, we numerically compare the closed-loop policy that we propose, to an open-loop and a static policy, to evaluate the value of responsiveness and of novelty respectively.

Our results open a number of questions for further research. First, the model can be tested empirically for validation and further refining. Second, we have taken the review period as fixed, e.g., a period is one week. This could in fact be a choice of the retailer: does it have to update the assortment weekly, knowing that this will create a significant workload for the organization? Or could it do it once a month only? This dilemma can be studied with our model, by solving the problem for different review cycles. Moreover, if the review cycle can also be determined dynamically, one could introduce a set-up cost associated with updating the assortment. Given the connection of our model to inventory problems, this seems a tractable extension to study, and it is likely that, for a single category, the optimal assortment policy is an (s, S) policy, where effort should only be invested if attractiveness is below s, in which case it should be brought up to S. Third, more strategic decisions can be considered. Namely, in our model responsiveness is fixed, i.e., the retailer can update the assortment with a zero lead-time, and we discuss when this can be done with positive lead-time. In practice, such responsiveness usually comes at a high cost, and Fisher (1997) suggests that should only be used when products are innovative, with high uncertainty (decay uncertainty in our case). If it is possible to adjust the assortment immediately at a high cost, or do it with a longer lead-time at a lower cost, what would the optimal combination be? How often and when should the adjustments be done at the last minute? This seems a difficult question, but it would shed light on a very relevant issue that retailers face.

Acknowledgements

V. Martínez-de-Albéniz's research was supported in part by the European Research Council - ref. ERC-2011-StG 283300-REACTOPS and by the Spanish Ministry of Economics and Competitiveness (Ministerio de Economía y Competitividad) - ref. ECO2011-29536.

References

Ainslie, A., X. Dreze, and F. Zufryden. 2005. Modeling movie life cycles and market share. *Marketing Science* 24 (3): 508–517.

Balakrishnan, A., M. Pangburn, and E. Stavrulaki. 2004. Stack Them High, Letem Fly: Lot-Sizing Policies When Inventories Stimulate Demand. *Management Science* 50 (5): 630–644.

Baucells, M., and R. Sarin. 2012. *Engineering happiness: A new approach for building a joyful life*. University of California Pr.

Bernstein, F., A. G. Kök, and L. Xie. 2010. Dynamic Assortment Customization with Limited Inventories. Working paper, Duke University.

Beyer, D., S. Sethi, and R. Sridhar. 2002. *Decision and control in management sciences in honor of professor alain haurie*, Chapter Average-Cost Optimality of a Base-Stock Policy for a Multi-Product Inventory Model with Limited Storage, 241–260. Kluwer Academic Publishers, Boston.

Beyer, D., S. P. Sethi, and R. Sridhar. 2001, December. Stochastic multiproduct inventory models with limited storage. *Journal of Optimization Theory and Applications* 111 (3): 553–588.

Blackburn, J., and G. Scudder. 2009. Supply chain strategies for perishable products: the case of fresh produce. *Production and Operations Management* 18 (2): 129–137.

Cachon, G., and M. Lariviere. 2005. Supply chain coordination with revenue-sharing contracts: Strengths and limitations. *Management Science* 51:30–44.

Caldentey, R., and F. Caro. 2010. Dynamic Assortment Planning. Working paper, UCLA.

Caro, F., and J. Gallien. 2007, February. Dynamic assortment with demand learning for seasonal consumer goods. *Management Science* 53 (2): 276–292.

Caro, F., J. Gallien, M. Diaz, J. Garcia, J. Manuel Corredoira, M. Montes, J. Antonio Ramos, and J. Correa. 2010, January. Zara Uses Operations Research to Reengineer Its Global Distribution Process. *Interfaces* 40 (1): 71–84.

Caro, F., and V. Martínez-de-Albéniz. 2009. The effect of assortment rotation on consumer choice and its impact on competition. In *Operations Management Models with Consumer-Driven Demand*, ed. S. Netessine and C. Tang. Springer US.

Caro, F., and V. Martínez-de-Albéniz. 2012, July. Product and Price Competition with Satiation Effects. *Management Science* 58 (7): 1357–1373.

Caro, F., V. Martínez-de-Albéniz, and P. Rusmevichientong. 2014. The Assortment Packing Problem: Multiperiod Assortment Planning for Short-Lived Products. *Management Science* Forthcoming:N/A.

Corstjens, M., and P. Doyle. 1981. A Model for Optimizing Retail Space Allocations. *Management Science* 27 (7): 822–833.

Curhan, R. C. 1973. Shelf Space Allocation and Profit Maximization in Mass Retailing. *Journal of Marketing* 37:54–60.

Esprit 2011. Annual report FY2010/2011. p.14.

Fisher, M. L. 1997. What is the right supply chain for your product? *Harvard Business Review* 75:105–117.

Honhon, D., V. Gaur, and S. Seshadri. 2010. Assortment planning and inventory decisions under stockout-based substitution. *Operations Research* 58 (5): 1364–1379.

Honhon, D., and A. G. Kök. 2011. Assortment planning with variety-seeking customers. Working paper, University of Texas at Austin.

Kök, A. G., M. Fisher, and R. Vaidyanathan. 2009. Assortment Planning: Review Of Literature And Industry Practice. *Retail Supply Chain Management: Quantitative Models And Empirical Studies* 122:99–153.

Krider, R. E., and C. B. Weinberg. 1998, February. Competitive dynamics and the introduction of new products: The motion picture timing game. *Journal of Marketing Research* 35 (1): 1–15.

Krishnan, H., R. Kapuscinski, and D. Butz. 2004. Coordinating contracts for decentralized supply chains with retailer promotional effort. *Management S* 50:48–63.

Lago, A., V. Martınez-de Albéniz, P. Moscoso, and A. Vall. 2013. The Role of Quick Response in Accelerating Sales of Fashion Goods. Working paper, IESE Business School.

Nahmias, S. 1982. Perishable inventory theory: A review. *Operations Research* 30 (4): 680–708.

Rusmevichientong, P., Z. J. M. Shen, and D. B. Shmoys. 2010, November. Dynamic Assortment Optimization with a Multinomial Logit Choice Model and Capacity Constraint. *Operations Research* 58 (6): 1666–1680.

Rusmevichientong, P., and H. Topaloglu. 2012, July. Robust Assortment Optimization in Revenue Management Under the Multinomial Logit Choice Model. *Operations Research* 60 (4): 865–882.

Sauré, D., and A. Zeevi. 2013. Optimal dynamic assortment planning with demand learning. *Manufacturing & Service Operations Management* n/a:Forthcoming.

Schaible, S., and J. Shi. 2003. Fractional programming: the sum-of-ratios case. *Optimization Methods and Software* 18 (2): 219–229.

Smith, J. E., and K. F. McCardle. 2002, September. Structural properties of stochastic dynamic programs. *Operations Research* 50 (5): 796–809.

Smith, S. A., and D. D. Achabal. 1998, March. Clearance pricing and inventory policies for retail chains. *Management Science* 44 (3): 285–300.

Talluri, K., and G. van Ryzin. 2004, January. Revenue management under a general discrete choice model of consumer behavior. *Management Science* 50 (1): 15–33.

Taylor, T. 2002. Supply chain coordination under channel rebates and sales effort effects. *Management Science* 48:992–1007.

Topkis, D. 1998. *Supermodularity and complementarity*. Princeton, NJ: Princeton University Press.

Ulu, C., D. Honhon, and A. Alptekinoglu. 2012, July. Learning Consumer Tastes Through Dynamic Assortments. *Operations Research* 60 (4): 833–849.

Veinott, A. F. 1965. Optimal Policy For A Multi-product, Dynamic, Nonstationary Inventory Problem. *Management Science* 12 (3): 206–222.

Wang, Y., and Y. Gerchak. 2001. Supply Chain Coordination when Demand Is Shelf-Space Dependent. *Manufacturing & Service Operations Management* 3 (1): 82–87.

Wei, Y., and Y. F. Chen. 2011, December. Joint determination of inventory replenishment and sales effort with uncertain market responses. *International Journal of Production Economics* 134 (2): 368–374.

Appendix of "A Closed-loop Approach to Dynamic Assortment Planning": Proofs

Proposition 1

Proof. We prove it by straightforward induction on t. R in Equation (3) is jointly concave in $\vec{y}_t$ if margins are identical across categories, i.e., $p_1 = \ldots = p_N$. When this is the case, the reward function $R(\vec{y}_t) - C(\vec{y}_t - \vec{x}_t)$ is jointly concave in $\vec{x}_t$ and $\vec{y}_t$. In addition, $\mathbb{E}[J_{t+1}(\vec{x}_{t+1})]$ is clearly jointly concave in $\vec{y}_t$ because $x_{it+1} = \varepsilon_{it+1} y_{it}$ (hence for each sample path, this is a concave function). The conclusion follows from Proposition 5 in Smith and McCardle (2002). Finally, $J_t(\vec{x}_t)$ is increasing with $\vec{x}_t$ because, given $\vec{u}_t = \vec{y}_t - \vec{x}_t$, $R(\vec{x}_t + \vec{u}_t)$ and $J_{t+1}(\vec{x}_{t+1})$ are increasing in $\vec{x}_t$. □

Theorem 1

Proof. Let us first assume that there is no capacity constraint. Taking the derivative of G_t in Equation (4) with respect to y_t we have the following optimality condition:

$$\frac{p}{(1+y_t)^2} + \beta \mathbb{E}[\varepsilon J'_{t+1}(\varepsilon y_t)] - c + \lambda_t = 0$$

Let b_t be the solution of this equation with $\lambda_t = 0$, which is the case for $x_t = 0$, for example. Then clearly if the constraint is not binding, $y_t^* = b_t$; otherwise, $\lambda_t > 0$ and $y_t^* = x_t > b_t$.

In addition, J'_t can be written as follows:

$$J'_t(x_t) = \min\left\{c, \frac{p}{(1+x_t)^2} + \beta \mathbb{E}[\varepsilon J'_{t+1}(\varepsilon x_t)]\right\}.$$

This comes from the fact that, given x_t, if an action is taken to increase x_t to $y_t \geq x_t$ then $J'_t(x_t)$ is equal to c, because the optimality condition yields $\frac{p}{(1+y_t)^2} + \beta \mathbb{E}[\varepsilon J'_{t+1}(\varepsilon y_t)] - c = 0$. If no action is taken $y_t = x_t$, then $\lambda_t > 0$, which gives $\frac{p}{(1+x_t)^2} + \beta \mathbb{E}[\varepsilon J'_{t+1}(\varepsilon x_t)] < c$ and $J'_t(x_t) = \frac{p}{(1+x_t)^2} + \beta \mathbb{E}[\varepsilon J'_{t+1}(\varepsilon x_t)]$.

We show by induction on t that $J'_t \geq J'_{t+1}$ and $b_t \geq b_{t+1}$. For period T, this is true, because $b_{T+1} = 0$ (no revenue after $T+1$ and positive cost) and

$$J'_T(x) \quad = \quad min\left\{c, \frac{p}{(1+x)^2}\right\} \geq 0.$$

Assuming that $J'_{t+2} \leq J'_{t+1}$, for any y, b_t, b_{t+1} satisfy $c = \frac{p}{(1+y)^2} + \beta\mathbb{E}[\varepsilon J'_{t+1}(\varepsilon y)]$, $\frac{p}{(1+y)^2} + \beta\mathbb{E}[\varepsilon J'_{t+2}(\varepsilon y)]$, and $b_t \geq b_{t+1}$ because the right-hand side functions are decreasing in y. It also follows that $J'_{t+1} \leq J'_t$ from the following definitions:

$$\begin{aligned} J'_t(x) &= min\Big\{c, \frac{p}{(1+x)^2} + \beta\mathbb{E}[\varepsilon J'_{t+1}(\varepsilon x)]\Big\} \\ J'_{t+1}(x) &= min\Big\{c, \frac{p}{(1+x)^2} + \beta\mathbb{E}[\varepsilon J'_{t+2}(\varepsilon x)]\Big\}. \end{aligned}$$

This completes the induction. If we add the capacity constraint, $a \geq y_t$, because of the convexity of the problem the optimal assort-up-to level will be $min(b_t, a)$. □

Proposition 2

Proof. Let us first assume that there is no capacity constraint. The first-order condition for a single period problem with $x = 0$ is:

$$\frac{p}{(1+b)^2} - c = 0.$$

From this equation we have $b = \varphi^1 := \sqrt{\frac{p}{c}} - 1$.

For the infinite-horizon problem, since ε is stationary, the assort-up-to level will be same for two consecutive periods. We can write the partial derivative as follows:

$$\frac{p}{(1+b)^2} + \beta\mathbb{E}[\varepsilon J'_{\infty}(\varepsilon b)] - c = 0 \tag{9}$$

Equation (9) gives the optimal assort-up-to level b. When the attractiveness level falls to εb, an order will be placed in the following period to reach the optimal level, thus we have that $J'_{\infty}(\varepsilon b) = c$. Hence, $\frac{p}{(1+b)^2} + \beta\bar{\varepsilon}c = c$, which yields $b = \varphi^{\infty} := \sqrt{\frac{p}{c(1-\beta\bar{\varepsilon})}} - 1$.

If we add the capacity constraint, $a \geq y_t$, because of the convexity of the problem the optimal assort-up-to level will be $min(b_t, a)$. □

Proposition 3

Proof. Let us first assume that there is no capacity constraint. Take $t_1 < t_2$. The sum in (6) is smaller for t_2 than for t_1. Indeed, for smaller t_1, more positive terms are added to the sum. Equating these different sums to c we get $\hat{b}_{t_1} \leq \hat{b}_{t_2}$ since the sum is a decreasing function of b. This proves the decreasing nature of $\hat{b}_t$ and yields the existence of t_{last}.

We prove the result for b_t by induction: it is equal to $\hat{b}_t$ if $t > t_{last}$, in which case $J'_t(x) = \min\left\{c, \sum_{\tau=0}^{T-t} \frac{(\beta\bar{\varepsilon})^\tau}{(1+\bar{\varepsilon}^\tau x)^2}\right\}$; or φ^∞ otherwise. For $t = T$, b_T satisfies

$$\frac{p}{(1+b)^2} = c,$$

i.e., $b_T = \hat{b}_T$. Moreover for $x \geq b_T$, $J'_T(x) = \frac{p}{(1+x)^2}$.

For $t+1 > t_{last}$, we thus have from the induction hypothesis that $b_{t+1} = \hat{b}_{t+1} < \varphi^\infty$. This implies that for $\bar{\varepsilon}b \leq b_{t+1} = \hat{b}_{t+1}$, Equation (5) can be written as

$$\frac{p}{(1+b)^2} + \beta\bar{\varepsilon}c = c,$$

i.e., $b = \varphi^\infty$. Hence if $\bar{\varepsilon}\varphi^\infty \leq \hat{b}_{t+1}$, then $t = t_{last}$ and $b_t = \varphi^\infty$. Otherwise, the equation above has no solution and thus we must look for a solution of Equation (5) in the range $\bar{\varepsilon}b > b_{t+1} = \hat{b}_{t+1}$. In this case, the inductive hypothesis yields

$$\frac{p}{(1+b)^2} + p\beta\bar{\varepsilon}\sum_{\tau=0}^{T-t-1} \frac{(\beta\bar{\varepsilon})^\tau}{(1+\bar{\varepsilon}^\tau b)^2} = p\sum_{\tau=0}^{T-t} \frac{(\beta\bar{\varepsilon})^\tau}{(1+\bar{\varepsilon}^\tau b)^2} = c,$$

and thus $b_t = \hat{b}_t$ and the inductive hypothesis is proven for t.

Finally when $t+1 \leq t_{last}$, we have that $b_{t+1} = \varphi^\infty < \hat{b}_{t+1}$. Hence Equation (5) has a solution at φ^∞, and thus $b_t = \varphi^\infty$, as well. If we add the capacity constraint, $a \geq y_t$, because of the convexity of the problem the optimal assort-up-to level will be $min(b_t, a)$. □

Theorem 2

Proof. The result follows from Theorem 3.3 in Beyer et al. (2001) where the authors study a multi-product inventory problem with stochastic demands and a warehousing constraint under discounted-cost criterion. They show that a modified base-stock policy is optimal. With this policy there exists a sequence of base-stock levels S_{ik} for product i in period k, such that if the beginning inventory position is below $\vec{S_k}$ then the optimal policy is to order up to $\vec{S_k}$, where $\vec{S_k}$ is the vector for base-stock levels for all products. On the other hand, when the beginning surplus is not below $\vec{S_k}$, a more complicated ordering policy is optimal. Under this policy, if a product is above its base-stock level, it may still be ordered or if it is below its base-stock level it might not be ordered. This type of feedback policies is established under the following assumptions which are satisfied for our problem: $G_t(\vec{y}_t)$ is continuous and $G_t(\vec{y}_t) \to -\infty$ as $\vec{y}_t \to \infty$. And from Proposition 1 $J_t(\vec{x}_t)$ is increasing and concave. $\square$

Proposition 4

Proof. For $T = 1$ (the single-period problem), the first-order conditions defined in (7) take the following form for product i, given $x_i = 0$:

$$\frac{p}{(1 + \sum_{j=1}^{N} b_j)^2} = c_i - \lambda_i.$$

Note that the left hand side would be the same for all products. Then λ_i could be equal to zero only for the product with the smallest cost. Let $i = arg\min_j c_j$, then $c_i = c_j - \lambda_j$ for $j \neq i$, which implies that only the assort-up-to level for i is positive. From the FOC, it is equal to $b_i = \varphi_i^1 = \sqrt{\frac{p}{c_i}} - 1$. For $k \neq i$, $b_k = 0$.

For the infinite-horizon with $T = \infty$, a stationary solution is optimal since we are considering a stationary problem. We have for all t (we omit t for brevity):

$$J(\vec{x}) = \max_{\vec{y} \geq \vec{x}} G(\vec{y}) = \sum_{i=1}^{N} c_i x_i + \sum_{i=1}^{N} \left(\frac{p y_i}{1 + \sum_{j=1}^{N} y_j} - c_i y_i \right) + \beta \mathbb{E}[J(\vec{\varepsilon} \vec{y})].$$

At optimality, the following equations must be satisfied for $\vec{b}$, starting at $x_j = 0$ for all j:

$$0 = \frac{p}{(1 + \sum_{j=1}^{N} b_j)^2} + \beta \mathbb{E} \left[\varepsilon_i \frac{\partial J}{\partial x_i} (\vec{\varepsilon} \vec{b}) \right] - c_i + \lambda_i. \tag{10}$$

We can write $\frac{\partial J(\vec{x})}{\partial x_i}$ in a similar way as we did for the single-product problem:

$$\frac{\partial J(\vec{x})}{\partial x_i} = \min \left\{ c_i, \frac{p}{(1 + \sum_{j=1}^{N} b_j)^2} + \beta \mathbb{E} \left[\varepsilon_i \frac{\partial J}{\partial x_i} (\vec{\varepsilon} \vec{b}) \right] \right\}.$$

If $b_i > 0$, the minimum is achieved at c_i, we have $\lambda_i = 0$ and thus,

$$\frac{p}{(1 + \sum_{j=1}^{N} b_j)^2} = (1 - \beta \bar{\varepsilon}_i) c_i.$$

Hence for $k \neq i$, unless $c_i(1 - \beta \bar{\varepsilon}_i) = c_k(1 - \beta \bar{\varepsilon}_k)$, λ_k and λ_i cannot be both equal to zero, which would imply a positive assort-up-to level for both i and k. As a result, b_i is positive only for i that satisfies $i = arg\min_j c_j(1 - \beta \bar{\varepsilon}_j)$. Then $b_i = \varphi_i^\infty = \sqrt{\frac{p}{c_i(1 - \beta \bar{\varepsilon}_i)}} - 1$. Otherwise, $b_k = 0$ for $k \neq i$. $\square$

Proposition 5

Proof. We proceed by induction.

For $t = T$, from Proposition 4 we know that for the single-period problem an investment is made only to the cheapest product. Let i be the product with the smallest cost. Then for all j, $\frac{\partial J_T}{\partial x_{jT}}(\bar{\varepsilon}\vec{y}_T) = min\left\{c_i, \frac{p}{(1+\sum_{j=1}^N x_{jT})^2}\right\}$, which depends only on the sum of x_{jT}, $j = 1, \ldots, N$ (an order is given for i if $\sum_{j=1}^N x_{jT} < b_{iT}$).

Now let us assume that this is true for all periods $t+1, \cdots, T$. For t, the FOC in (7) implies:

$$c_j - \lambda_{jt} = \frac{p}{(1+\sum_{j=1}^N b_{jt})^2} + \beta\bar{\varepsilon}\frac{\partial J_{t+1}}{\partial x_{jt+1}}(\bar{\varepsilon}\vec{b}_t).$$

In the periods after t, only investments into product i are possible, and thus J_{t+1} depends on b_{it+1} only: if the sum $\sum_{j=1}^N x_{jt+1} < b_{it+1}$ an investment will be made, and no action will be taken otherwise. Hence J_{t+1} will be a function of c_i, b_{it+1} and $\sum_{j=1}^N x_{jt+1}$ only, which again makes it the same for all j. This means $c_j - \lambda_{jt}$ is the same for all j. With the same argument as before, an investment is made only to product i in period t. □

Proposition 6

Proof. In the last period $t = 2$, investment will only take place on category 1, per Proposition 4. Thus for $i = 1, 2$, $\frac{\partial J_T}{\partial x_{iT}}(\vec{x}_T) = \min\left\{c_1, \frac{1}{(1+x_{1T}+x_{2T})^2}\right\}$.

FOCs for b_{1t} and b_{2t} for $t = 1$, when they are both starting from zero level of attractiveness and they are both positive, are given as follows:

$$\frac{1}{(1+b_{1t}+b_{2t})^2} + \bar{\varepsilon}_1 \min\left\{c_1, \frac{1}{(1+\bar{\varepsilon}_1 b_{1t}+\bar{\varepsilon}_2 b_{2t})^2}\right\} = c_1 \tag{11}$$

$$\frac{1}{(1+b_{1t}+b_{2t})^2} + \bar{\varepsilon}_2 \min\left\{c_1, \frac{1}{(1+\bar{\varepsilon}_1 b_{1t}+\bar{\varepsilon}_2 b_{2t})^2}\right\} = c_2. \tag{12}$$

We first show that there is no investment in $t = 2$ if both assort-up-to levels are positive in $t = 1$. Indeed, if the investment was positive, then the equations above would imply that $(1-\bar{\varepsilon}_1)c_1 = c_2 - \bar{\varepsilon}_2 c_1$. This is a degenerate case that also yields a solution where only investment into one category at $t = 1$. Hence the minimum in (11) and (12) is equal to $\frac{1}{(1+\bar{\varepsilon}_1 b_{1t}+\bar{\varepsilon}_2 b_{2t})^2}$. b_{1t}

and b_{2t} are the unique solutions to the following equations from (11) and (12):

$$\frac{1}{(1+b_{1t}+b_{2t})^2} = c_1 - \frac{\bar{\varepsilon}_1}{(1+\bar{\varepsilon}_1 b_{1t}+\bar{\varepsilon}_2 b_{2t})^2}$$
$$\frac{\bar{\varepsilon}_1 - \bar{\varepsilon}_2}{(1+\bar{\varepsilon}_1 b_{1t}+\bar{\varepsilon}_2 b_{2t})^2} = c_1 - c_2,$$

which gives:

$$b_{1t} + b_{2t} = \sqrt{\frac{1}{c_1 - \bar{\varepsilon}_1\left(\frac{c_1-c_2}{\bar{\varepsilon}_1-\bar{\varepsilon}_2}\right)}} - 1 = \theta_2 - 1 \tag{13}$$
$$\bar{\varepsilon}_1 b_{1t} + \bar{\varepsilon}_2 b_{2t} = \sqrt{\frac{1}{\frac{c_1-c_2}{\bar{\varepsilon}_1-\bar{\varepsilon}_2}}} - 1 = \theta_1 - 1. \tag{14}$$

Solving these equations we get:

$$b_{11} = \frac{\theta_1 - \theta_2\bar{\varepsilon}_2 + (\bar{\varepsilon}_2 - 1)}{\bar{\varepsilon}_1 - \bar{\varepsilon}_2}$$
$$b_{21} = \frac{\theta_2\bar{\varepsilon}_1 - \theta_1 + (1 - \bar{\varepsilon}_1)}{\bar{\varepsilon}_1 - \bar{\varepsilon}_2}$$

with $\theta_1 = \sqrt{\frac{\bar{\varepsilon}_1-\bar{\varepsilon}_2}{c_1-c_2}}$ and $\theta_2 = \sqrt{\frac{1}{c_1-\bar{\varepsilon}_1/\theta_1^2}}$. Note that for (13) to be consistent we must have $\frac{c_1-c_2}{\bar{\varepsilon}_1-\bar{\varepsilon}_2} \leq c_2$. Hence we have the following conditions to be satisfied along with the condition $c_1/\bar{\varepsilon}_1 < c_2/\bar{\varepsilon}_2$, which says $\bar{\varepsilon}_1$ must be large enough to compensate the extra cost to invest in product 1:

$$0 \leq \frac{c_1 - c_2}{\bar{\varepsilon}_1 - \bar{\varepsilon}_2} \leq c_2 \leq c_1.$$

□

Proposition 7

Proof. We proceed by induction.

For $t = T$, from the FOC, for category i:

$$c - \lambda_{iT} = \frac{p}{(1+\sum_{j=1}^{N} b_{jT})^2}.$$

As a result, one optimal solution (out of many) is to set $b_{iT} = \frac{1}{N}\left(\sqrt{\frac{p}{c}} - 1\right)$.

Now let us assume that this is true for all periods $t+1, \cdots, T$, that is, assort-up-to levels are equal for all products. For t, the FOC implies:

$$c - \lambda_{it} = \frac{p}{(1+\sum_{j=1}^{N} b_{jt})^2} + \beta \mathbb{E}\left[\varepsilon_i \frac{\partial J_{t+1}}{\partial x_{it+1}}(\vec{\varepsilon b_t})\right].$$

Denote the expectation in the right hand side $h_i(b_{1t}, b_{2t}, \cdots, b_{Nt})$. Since the decays are i.i.d. we can show that for example for product 1 and product 2 $h_1(b_{1t}, b_{2t}, \cdots, b_{Nt}) = h_2(b_{2t}, b_{1t}, \cdots, b_{Nt})$. If the assort-up-to levels are positive for products 1 and 2 the following is true:

$$\begin{aligned} c &= \frac{p}{(1+\sum_{j=1}^{N} b_{jt})^2} + \beta h_1(b_{1t}, b_{2t}, \cdots, b_{Nt}) \\ c &= \frac{p}{(1+\sum_{j=1}^{N} b_{jt})^2} + \beta h_2(b_{1t}, b_{2t}, \cdots, b_{Nt}), \end{aligned}$$

which implies $h_1(b_{1t}, b_{2t}, \cdots, b_{Nt}) = h_1(b_{1t}, b_{2t}, \cdots, b_{Nt})$, which could not be true if $b_{1t} \neq b_{2t}$. Note that this is true for any two products, which completes the induction. □

Proposition 8

Proof. For the single-period problem, with $x_i = 0$, the optimization problem can be written as

$$\max_{\vec{b} \geq 0} \frac{\sum_{i=1}^{N} p_i b_i}{1+\sum_{i=1}^{N} b_i} - \sum_{i=1}^{N} c_i b_i.$$

Letting $z = \sum_{i=1}^{N} b_i$ and $\theta_i = b_i / z$, this is equivalent to

$$\max_{z \geq 0} \max_{\vec{\theta} \geq 0 | \sum_{i=1}^{N} \theta_i = 1} \frac{z \sum_{i=1}^{N} p_i \theta_i}{1+z} - \sum_{i=1}^{N} c_i z \theta_i.$$

Given z, the second maximization program is linear and yields an optimal solution where there is i such that $\theta_i = 1$ and $\theta_j = 0$ for $j \neq i$. This proves that at optimality, investment should only occur in one category. To determine which one, it is sufficient to consider the optimal single-category decisions, i.e., $b_i = \sqrt{\frac{p_i}{c_i}} - 1 \geq 0$. This yields a profit of

$$\frac{p_i b_i}{1+b_i} - c_i b_i = \frac{p_i \sqrt{\frac{p_i}{c_i}} - p_i}{\sqrt{\frac{p_i}{c_i}}} - c_i \sqrt{\frac{p_i}{c_i}} + c_i = p_i - 2\sqrt{p_i c_i} + c_i = (\sqrt{p_i} - \sqrt{c_i})^2.$$

Thus the category with the largest $\sqrt{p_i} - \sqrt{c_i} \geq 0$ should be chosen.

For $T = \infty$, the same derivation can be used replacing cost by $c_i(1 - \beta\bar{\varepsilon}_i)$. $\square$

Proposition 9

Proof. For $T = 1$, the first-order conditions for the capacitated problem take the following form for category i, given $x_i = 0$:

$$\frac{p}{(1 + \sum_{j=1}^{N} b_j)^2} = c_i + \lambda_{i1} - \lambda_{i2}.$$

where λ_{i1} and λ_{i2} are the KKT multipliers associated with the constraints $a_i \geq b_i$ and $b_i \geq 0$, respectively. The possible values that the right hand side can take are $c_i - \lambda_{i2}$ if no order is given, $c_i + \lambda_{i1}$ if an order is given with $b_i = a_i$, and c_i if and order is given with $b_i < a_i$. Note that the left hand side is the same for all categories, hence for a category i_k, $b_{i_k} < a_{i_k}$ is possible either if $c_{i_k} = \frac{p}{(1+b_{i_k})^2}$, which implies $b_{i_k} = \sqrt{\frac{p}{c_{i_k}}} - 1$, or when the cost for category i_k satisfies $c_{i_k} > c_j$ for $j \in \{1, \cdots, k-1\}$ with $b_{i_j} = a_{i_j}$, $\forall j$, and $c_{i_k} = \frac{p}{(1+\sum_{j=1}^{k} b_{i_j})^2}$. The latter is the case when b_{i_1} was raised as high as possible, but the optimum sum for the attractiveness is not achieved. In this case categories are added in the assortment in increasing cost to achieve better total attractiveness level until for category i_k, $\hat{b}_{i_k} < a_{i_k}$ is satisfied, which is equal to saying that $c_{i_k} = \frac{p}{(1+\sum_{j=1}^{k} b_{i_j})^2}$. In the case this equality is not satisfied for any i, $\varphi_{i_k}^1 - \sum_{j=1}^{k-1} a_{i_j} < 0$, which means at the optimum all categories are raised to their respective maximum possible attractiveness levels.

For the infinite-horizon with $T = \infty$, a stationary solution is optimal since we are considering a stationary problem. We have

$$J(\vec{x}) = \max_{\vec{y} \geq \vec{x}} G(\vec{y}) = \sum_{i=1}^{N} c_i x_i + \sum_{i=1}^{N} \left(\frac{p y_i}{1 + \sum_{j=1}^{N} y_j} - c_i y_i \right) + \beta \mathbb{E}[J(\vec{\varepsilon} \vec{y})].$$

At optimality, the following equations must be satisfied:

$$0 = \frac{p}{(1 + \sum_{j=1}^{N} b_j)^2} + \beta \mathbb{E}\left[\varepsilon_i \frac{\partial J}{\partial x_i}(\vec{\varepsilon}\vec{b})\right] - c_i - \lambda_{i1} + \lambda_{i2} \tag{15}$$

with

$$\frac{\partial J(\vec{x})}{\partial x_i} = \frac{p}{(1 + \sum_{j=1}^{N} b_j)^2} + \beta \mathbb{E}\left[\varepsilon_i \frac{\partial J}{\partial x_i}(\vec{\varepsilon}\vec{b})\right] - \lambda_{i1} + \lambda_{i2}.$$

Similar to the single period case, for category i_k, $b_{i_k} < a_{i_k}$ is possible either if $c_{i_k}(1 - \beta\bar{\varepsilon}_{i_k}) =$

$\frac{p}{(1+b_{i_k})^2}$ or when category i_k satisfies $c_{i_k}(1-\beta\bar{\varepsilon}_{i_k}) > c_{i_j}(1-\beta\bar{\varepsilon}_{i_j})$ for all categories i_j with $b_{i_j} = a_{i_j}$. The first is true since if the order is less than a_{i_k} in the stationary case it will continue to be similar in the following periods, hence we can replace $\frac{\partial J(\vec{x})}{\partial x_{i_k}}$ in the expectation by c_{i_k} and then an investment is made only to this category in the optimum. If this is not true there should exist categories i_j with $b_{i_j} = a_{i_j}$, for which the behavior of the optimum is again similar for the following periods and by replacing $\frac{\partial J(\vec{x})}{\partial x_{i_j}}$ for i_j we can show that $c_{i_k}(1-\beta\bar{\varepsilon}_{i_k}) > c_{i_j}(1-\beta\bar{\varepsilon}_{i_j})$. In this case categories are added in the assortment in increasing $c_{i_j}(1-\beta\bar{\varepsilon}_{i_j})$ values, hence a similar result to $T = 1$ follows by replacing c_i with $c_i(1-\beta\bar{\varepsilon}_i)$.

□

Proposition 10

Proof. The derivative of the objective function for the unequal margin problem can be written as follows given K and I:

$$\frac{-\sum_{k\in K} a_k p_k - (z - \sum_{k\in K} a_k)p_i}{(1+z)^2} + \frac{p_i}{1+z} - c_i = \frac{-\sum_{k\in K} a_k(p_k - p_i) + p_i}{(1+z)^2} - c_i.$$

Note that this derivative does not change as z is infinitesimally altered at any point, but it does change as the set K and possibly I change as the order of d_js changes. This change occurs as z changes in pairs, that is, every time a change occurs it involves two categories. There are three possibilities: first d_js for two categories that are not in the current K change, which obviously does not change the derivative. Second, d_ks for two categories that are in the current K with both $b_k = a_k$ can change, again this does not change the derivative. Third, the smallest d_k in the current K, d_{k_i}, and d_i might be changing orders. In the latter case the derivative will be changing from

$$\frac{-\sum_{k\in K} a_k p_k - (z - \sum_{k\in K} a_k)p_i}{(1+z)^2} + \frac{p_i}{1+z} - c_i$$

to

$$\frac{-\sum_{k\in K\backslash k_i} a_k p_k - a_i p_i - (z - a_i - \sum_{k\in K\backslash k_i} a_k)p_{k_i}}{(1+z)^2} + \frac{p_{k_i}}{1+z} - c_{k_i}.$$

Since at the intersection point $d_i = d_{k_i}$, that is, $\frac{p_i}{1+z} - c_i = \frac{p_{k_i}}{1+z} - c_{k_i}$, their difference is equal to $(p_{k_i} - p_i)(\sum_{k\in K} a_k - a_i - z)$, which is where the change occurs in the derivative.

From the definition of the derivative we see that, given K, we either have a concave or a convex function at hand, depending on the parameters a_ks and p_ks (specifically, on the sign of $e_K = -\sum_{k\in K} a_k(p_k - p_i) + p_i$). If we have concavity, that is, when $e_K > 0$, we evaluate for the solution with K and I. By evaluating all possible derivatives we can find out which solution gives the best z and hence the optimal categories to invest in depending on the capacities. The number of possible orderings for d_js is the number of intersections where d_js change order of magnitude as z increases plus 1, which is equal to $\frac{N(N-1)}{2} + 1$. Given an interval for z the possible solutions where we need to evaluate the derivative are limited to N. Note that for the case $(K, I) = (\emptyset, \{i_1\})$, where there is no binding capacity constraints, the single category in a given interval with the highest d_{i_1} is evaluated at an attractiveness level of $min(\varphi^1_{i_1}, UB)$. There is also the extreme case of a solution with a subset of categories only with binding constraints, where $I = \emptyset$ (N possible solutions). Hence, the maximum number of evaluations is $(\frac{N(N-1)}{2} + 1)N + N$. For $T = \infty$, the same algorithm can be used replacing c_j by $c_j(1 - \beta\bar{\varepsilon}_j)$ for all j.

□